Midterm Exam 136.151
BRIEF Solutions

Values

[12] 1. Evaluate the limit or show that it does not exist.

[4] (a) $\lim\limits_{x\to 2}\dfrac{\sqrt{x+2}-2}{x-2}$

$$\lim_{x\to 2}\frac{\sqrt{x+2}-2}{x-2}=\lim_{x\to 2}\frac{\sqrt{x+2}-2}{x-2}\frac{\sqrt{x+2}+2}{\sqrt{x+2}+2}=\lim_{x\to 2}\frac{x+2-4}{(x-2)(\sqrt{x+2}+2)}=\lim_{x\to 2}\frac{1}{(\sqrt{x+2}+2)}=\frac{1}{4}$$

[4] (b) $\lim\limits_{x\to\infty}(x-\sqrt{1+x^2})$

$$\lim_{x\to\infty}(x-\sqrt{1+x^2})=\lim_{x\to\infty}(x-\sqrt{1+x^2})\frac{x+\sqrt{1+x^2}}{x+\sqrt{1+x^2}}=\lim_{x\to\infty}\frac{x^2-1-x^2}{x+\sqrt{1+x^2}}=\lim_{x\to\infty}\frac{-1}{x+\sqrt{1+x^2}}=0,$$

since the denominator in the last limit obviously tends to infinity while the numerator is –1.

(Note: $\lim\limits_{x\to\infty}(x-\sqrt{1+x^2})=\infty-\infty=0$ or any variant of that false argument is worth 0 marks.)

[4] (c) $\lim\limits_{x\to 2}\dfrac{x-2}{|x-2|}$

$$\lim_{x\to 2^+}\frac{x-2}{|x-2|}=\lim_{x\to 2^+}\frac{x-2}{x-2}=\lim_{x\to 2^+}1=1,$$

$$\lim_{x\to 2^-}\frac{x-2}{|x-2|}=\lim_{x\to 2^-}\frac{x-2}{-(x-2)}=\lim_{x\to 2^+}(-1)=-1.$$

Since the two one-sided limits are not equal, the original limit does not exist.

Values

[7] 2. Let $f(x)=3x^2+x$. Find $f'(2)$ using **ONLY** the definition of the derivative.

$$f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{3(2+h)^2+(2+h)-14}{h}=$$

$$=\lim_{h\to 0}\frac{3(4+4h+h^2)+(2+h)-14}{h}=\lim_{h\to 0}\frac{13h+3h^2}{h}=\lim_{h\to 0}13+3h=13$$

[13] 3. Compute the following derivatives. Do NOT simplify your answer after differentiating.

[4] (a) $f'(x)$ if $f(x)=\sin(\sqrt{1+x^2})$

$$f'(x)=\cos(\sqrt{1+x^2})\frac{1}{2\sqrt{1+x^2}}2x.$$

[4] (b) $g''(x)$ if $g(x)=\dfrac{x}{1-x}$

$$g'(x)=\frac{(1-x)-x(-1)}{(1-x)^2}=\frac{1}{(1-x)^2}.$$

$$g''(x)=\left((1-x)^{-2}\right)'=-2(1-x)^{-3}(-1)=\frac{2}{(1-x)^3}.$$

[5] (c) $h'(x)$ if $h(x) = x^{(x-1)}$.

Start with $h(x) = x^{(x-1)}$ and apply the natural logarithm on both sides. Get $\ln h(x) = \ln x^{(x-1)}$, and so $\ln h(x) = (x-1)\ln x$. Now differentiate implicitly with respect to *x*: $\frac{1}{h(x)}h'(x) = \ln x + \frac{x-1}{x}$, so that $h'(x) = h(x)(\ln x + \frac{x-1}{x}) = x^{x-1}(\ln x + \frac{x-1}{x})$.

[8] 4. Let $f(x) = e^{\cos x}$

[4] (a) Find $f''(0)$.

$f'(x) = e^{\cos x}(-\sin x)$
$f''(x) = e^{\cos x}(-\sin x)(-\sin x) + e^{\cos x}(-\cos x)$
$f''(0) = 0 + e(-1) = -e$.

[4] (b) Find the equation of the tangent line of the curve $f(x) = e^{\cos x}$ at the point when $x = \frac{\pi}{2}$.

The slope of that tangent line is $f'(\frac{\pi}{2}) = e^{\cos\frac{\pi}{2}}(-\sin\frac{\pi}{2}) = -1$. So, the equation of the tangent is $y = (-1)x + b$. When $x = \frac{\pi}{2}$ we see that $f(\frac{\pi}{2}) = 1$. Since the line passes through that point we have that $1 = (-1)\frac{\pi}{2} + b$, from where we find that $1 + \frac{\pi}{2} = b$. So, the equation of the tangent line is $y = (-1)x + 1 + \frac{\pi}{2}$.

[10] 5. The equation $y^5 - y\cos x = 0$ defines *y* as a function on *x*.

[6] (a) Evaluate $\frac{dy}{dx}$ at the point (0,1).

Differentiate implicitly to get $5y^4\frac{dy}{dx} - (\frac{dy}{dx}\cos x + y(-\sin x)) = 0$. At the point (0,1) we get $5\frac{dy}{dx} - \frac{dy}{dx} = 0$ from where we find that $\frac{dy}{dx} = 0$

[4] (b) Find an equation of the tangent line to the curve $y^5 - y\cos x = 0$ at the point $(0,1)$.

Since, by part (a), the slope of the tangent is 0, its equation is $y = b$, where *b* is the *y*-intercept. But it is given that $b = 1$ and so the equation is $y = 1$.

[4] 6. [bonus] Suppose $f(x)$ and $g(x)$ are continuous at $x = a$. Show that the function $f(x)g(x)$ is also continuous at $x = a$.

$\lim_{x\to a} f(x)g(x) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) = f(a)g(a)$; in the first equality the property that we have used is applicable since the limits on the right-hand side exist; in the second equality we have used our assumptions for $f(x)$ and $g(x)$.