February 23, 2016

THE UNIVERSITY OF MANITOBA

MIDTERM EXAM

DEPARTMENT & COURSE NO: Math 1500

COVER PAGE

EXAMINATION: Intro. to Calculus TIME: 1 HOUR EXAMINER: Various

NAME: (Print in ink) Solutions

STUDENT NUMBER: ____________

SIGNATURE: (in ink) ____________

(I understand that cheating is a serious offense)

☐	A01	10:30-11:20 MWF	A. Clay
☐	A02	9:30-10:20 MWF	R. Borgersen
☐	A03	8:30-9:45 TR	A. Barria Comicheo
☐	A04	11:30-12:45 TR	M. Virgilio
☐	A05	13:00-14:15 TR	R. Borgersen
☐	A06	15:30-16:20 MWF	S. Kalajdzievski
☐	Challenge for credit		

INSTRUCTIONS TO STUDENTS:

This is a one hour exam. Show all your work and **justify** your answers. **Unjustified answers will receive LITTLE or NO CREDIT.**

No aids, calculators or other electronic devices of any kind are permitted during the examination.

This exam has a title page, 7 pages of questions and 1 blank page for rough work. Please check that you have all the pages. You may remove the blank page if you want, but be careful not to loosen the staples.

The value of each question is indicated beside the statement of the question. The total value of all questions is 60 points.

Answer all questions on the exam paper in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

Question	Points	Score
1	6	
2	12	
3	7	
4	7	
5	7	
6	10	
7	11	
Total	60	

[6] 1. [3] (a) Which of the following functions is even, which is odd, which is neither even nor odd? Justify your answers!

$f(x) = 2^x + 2^{-x}$ $\quad g(x) = x^3 - \sqrt{x}$ $\quad h(x) = x2^{|x|}$

$f(-x) = 2^{-x} + 2^{-(-x)} = 2^x + 2^{-x} = f(x)$ EVEN

$g(-x) = (-x)^3 - \sqrt{-x} \neq g(x), -g(x)$ NEITHER

$h(-x) = -x2^{|-x|} = -x2^{|x|} = -h(x)$ ODD

[3] (b) Find the domain of the function $\dfrac{\sqrt{x^2-3x}}{(x+3)(x-5)}$. Express your final answer in terms of intervals.

Problems if

$x^2 - 3x < 0$

$x(x-3) < 0$

$(x+3)(x-5) = 0$

$x = -3$ or 5

	$-\infty$ to 0	0 to 3	3 to ∞
x	$-$	$+$	$+$
$x-3$	$-$	$-$	$+$
$x(x-3)$	$+$	$-$	$+$

$\uparrow$

$x \in (0,3)$

Thus

Domain is

$(-\infty,-3) \cup (-3,0] \cup [3,5) \cup (5,\infty)$

[12] **2.** Calculate each of the limits (a), (b), and (c), if they exist. If the limit does not exist, determine whether the limit is ∞, $-\infty$ or neither.

[4] (a) $\lim_{x\to 1}\frac{x^2-3x+2}{x^2-1}$ $= \lim_{x\to 1}\frac{\cancel{(x-1)}(x-2)}{\cancel{(x-1)}(x+1)} =$

$= \lim_{x\to 1}\frac{x-2}{x+1} = \frac{-1}{2}$

[4] (b) $\lim_{x\to\infty}\frac{\sqrt{x^3-1}}{x\sqrt{x}}$ $= \lim_{x\to\infty}\frac{\cancel{\sqrt{x^3}}\sqrt{1-1/x^3}}{\cancel{x^{3/2}}}$

$= \lim_{x\to\infty}\sqrt{1-\cancel{\frac{1}{x^3}}^{\,0}} = 1.$

[4] (c) $\lim_{x\to 1}\left[(x-1)^2\cos\left(\frac{1}{x-1}\right)\right]$ [Hint: Squeeze theorem!]

$-1 \le \cos\left(\frac{1}{x-1}\right) \le 1$

$-(x-1)^2 \le (x-1)^2\cos\left(\frac{1}{x-1}\right) \le (x-1)^2$

$\lim_{x\to 1} -(x-1)^2 = 0$

$\lim_{x\to 1} (x-1)^2 = 0$

$\therefore$ By the squeeze theorem, $\lim_{x\to 1}(x-1)^2\cos\left(\frac{1}{x-1}\right) = 0.$

[7] 3. [2] (a) State the definition of continuity: what exactly does it mean to say that a function $f(x)$ is continuous at $x = a$?

$f(x)$ is continuous at $x=a$ if

$$\lim_{x\to a} f(x) = f(a)$$

[5] (b) Find the constant c such that the function $f(x) = \begin{cases} 2^{\sqrt{cx-1}} & \text{if } x \geq 1 \\ \dfrac{x^3-1}{3(x-1)} & \text{if } x < 1 \end{cases}$ is continuous when $x = 1$.

We need $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1) = 2^{\sqrt{c-1}}$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^3-1}{3(x-1)} = \lim_{x\to 1^-} \frac{\cancel{(x-1)}(x^2+x+1)}{3\cancel{(x-1)}}$$

$$= \lim_{x\to 1^-} \frac{x^2+x+1}{3} = 1.$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2^{\sqrt{cx-1}} = 2^{\sqrt{c-1}}$$

$$2^{\sqrt{c-1}} = 1 = 2^0$$

$$\sqrt{c-1} = 0$$

$$c-1 = 0$$

$$c = 1.$$

[7] 4. [2] (a) State the definition of differentiability: what exactly does it mean to say that a function $f(x)$ is differentiable at $x = a$.

f(x) is differentiable at x=a if

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

exists.

[5] (b) Use the definition of differentiability to find the derivative of the function $f(x) = \frac{1}{x-1}$ at the point when $x = 2$. No marks will be given if other methods are used.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h-1} - \frac{1}{x-1}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{(\cancel{x}-\cancel{1})-(\cancel{x}+h-\cancel{1})}{(x+h-1)(x-1)}\right)$$

$$= \lim_{h\to 0} \frac{1}{\cancel{h}}\left(\frac{-\cancel{h}}{(x+h-1)(x-1)}\right)$$

$$= \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}$$

$$f'(2) = \frac{-1}{(2-1)^2} = \frac{-1}{}$$

[7] **5.** Prove the product rule of differentiation: $\left(f(x)g(x)\right)' = f'(x)g(x) + f(x)g'(x)$.

[10] 6. **[7]** (a) If $yx + y^3 - x^2 = 1$, find $\frac{dy}{dx}$ at the point $(1,1)$.

$$\frac{d}{dx}\left(yx + y^3 - x^2\right) = \frac{d}{dx}(1)$$

$$y(1) + y'x + 3y^2y' - 2x = 0$$

@(1,1):

$$1 + y' + 3y' - 2 = 0$$

$$4y' - 1 = 0$$

$$y' = \frac{1}{4}$$

[3] (b) Find the equation of the tangent line to the curve $yx + y^3 - x^2 = 1$ at the point $(1,1)$.

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{1}{4}(x - 1)$$

$$y = \frac{1}{4}x - \frac{1}{4} + 1$$

$$y = \frac{1}{4}x + \frac{3}{4}$$

[11] 7. Find $\frac{dy}{dx}$. DO NOT SIMPLIFY YOUR ANSWER.

[3] (a) $y = 5x^8 + e^{3x} - \pi^4 + \sqrt{x^3}$

$$y' = 40x^7 + e^{3x}(3) - 0 + \frac{3}{2}x^{\frac{1}{2}}$$

[4] (b) $y = \frac{e^{(x^2)}}{\sqrt{x} - 1}$

$$y' = \frac{(\sqrt{x} - 1)(e^{(x^2)} \cdot 2x) - (e^{(x^2)})(\frac{1}{2}x^{-\frac{1}{2}})}{(\sqrt{x} - 1)^2}$$

[4] (d) $y = \underbrace{(\sin^3 x)}\underbrace{\cos(x^2)}$

$$y' = (\sin^3 x)(-\sin(x^2) \cdot 2x) + 3(\sin^2 x)(\cos x)(\cos(x^2))$$