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1. Compute the following limits, if they exist. If a limit does not exist, indicate whether the limit tends to ∞ or $-\infty$, if applicable; otherwise write "does not exist". Be sure to justify your responses.

[5] (a) $\displaystyle\lim_{x\to 4}\frac{\sqrt{x}-2}{2x-16}$.

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{2x-16} = \frac{\sqrt{4}-2}{2(4)-16} = \frac{0}{-8} = 0.$$

[5] (b) $\displaystyle\lim_{x\to\infty}\sqrt{x+1}-\sqrt{x}$.

$$\begin{aligned}\lim_{x\to\infty}\sqrt{x+1}-\sqrt{x} &= \lim_{x\to\infty}(\sqrt{x+1}-\sqrt{x})\cdot\frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}}\\ &= \lim_{x\to\infty}\frac{(x+1)-x}{\sqrt{x+1}+\sqrt{x}}\\ &= \lim_{x\to\infty}\frac{1}{\sqrt{x+1}+\sqrt{x}}\\ &= 0.\end{aligned}$$

[5] (c) $\displaystyle\lim_{y\to 1^-}\frac{1}{y-1}-\frac{1}{|y-1|}$.

$$\begin{aligned}\lim_{y\to 1^-}\frac{1}{y-1}-\frac{1}{|y-1|} &= \lim_{y\to 1^-}\frac{1}{y-1}-\frac{1}{-(y-1)}\\ &= \lim_{y\to 1^-}\frac{1}{y-1}+\frac{1}{y-1}\\ &= \lim_{y\to 1^-}\frac{2}{y-1}\\ &= -\infty.\end{aligned}$$

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2. Compute the derivative.
   DO NOT SIMPLIFY YOUR ANSWER.

[5] (a) Compute dy/dx if $y = 5/x + e^e - 45x^{\frac{2}{5}} + x^3$.

$$y = 5x^{-1} + e^e - 45x^{\frac{2}{5}} + x^3$$
$$\frac{dy}{dx} = -5x^{-2} + 0 - \frac{90}{5}x^{-\frac{3}{5}} + 3x^2.$$

[5] (b) Compute $f'(t)$ given $f(t) = \dfrac{t-1}{t+1} + \sin(2t)$.

$$f'(t) = \frac{(t+1)(1) - (t-1)(1)}{(t+1)^2} + \cos(2t) \cdot (2).$$

[5] (c) Compute $u'(s)$ when $u(s) = s\, e^{\cos(s)}$.

$$u'(s) = s \cdot e^{\cos(s)} \cdot (-\sin(s)) + (1)e^{\cos(s)}.$$

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[10] 3. Find the values of a and b for which the function f is continuous on its domain. Be sure to justify your answer.

$$f(x) = \begin{cases} a & \text{if } x < -1, \\ \dfrac{1}{x^2-3} & \text{if } -1 \le x \le 2, \\ \dfrac{x+1}{b} & \text{if } x > 2. \end{cases}$$

Since the 'pieces' of f are rational functions for any a and b, they are continuous on their domains. We only need to ensure continuity for $x = -1$ and $x = 2$.
We need

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) = f(-1)$$

and

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2).$$

- $\displaystyle\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} a = a$
- $\displaystyle\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \frac{1}{x^2-3} = \frac{-1}{2}.$
- $f(-1) = \dfrac{-1}{2}.$

Hence $a = -\frac{1}{2}$.

- $\displaystyle\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{1}{x^2-3} = 1$
- $\displaystyle\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{x+1}{b} = \frac{3}{b}.$
- $f(2) = 1.$

Hence $\frac{3}{b} = 1$, so $b = 3$.

Notes:

- The question asks for the values of a and b for which the function is continuous on its domain. Hence it is not enough to just find the values of a and b for which the function is continuous at $x = -1$ and $x = 2$. You should also explain why it is continuous everywhere else in its domain.

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[10] 4. **Use the definition** of the derivative to compute $f'(3)$ if

$$f(x) = \sqrt{x+1}.$$

$$\begin{aligned}
f'(3) &= \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{4+h}-2}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{4+h}-2}{h}\left(\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}\right) \\
&= \lim_{h\to 0} \frac{(4+h)-4}{h(\sqrt{4+h}+2)} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{4+h}+2)} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{4+h}+2} \\
&= \frac{1}{\sqrt{4+0}+2} \\
&= \frac{1}{4}.
\end{aligned}$$

Notes:

- Finding $f'(x)$ using the definition and then substituting $x = 3$ is also acceptable.

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[10] 5. Use the intermediate value theorem to prove that the equation

$$0 = (1+t)^{-2} - \sin(\pi t)$$

has a solution.

Let $f(t) = \dfrac{1}{(1+t)^2} - \sin(\pi t)$.

The function $f(t)$ is a difference of $\dfrac{1}{(1+t)^2}$, which is continuous on $(-\infty, -1) \cup (-1, \infty)$, and $\sin(\pi t)$, which is continuous on all of $\mathbb{R}$, and is therefore continuous on $(-\infty, -1) \cup (-1, \infty)$.
In particular, $f(t)$ is continuous on the interval $[0, \frac{1}{2}]$.

- $f(0) = 1 - 0 = 1 > 0$.
- $f(\frac{1}{2}) = \dfrac{1}{(\frac{3}{2})^2} - \sin(\frac{\pi}{2}) = \dfrac{4}{9} - 1 = -\dfrac{5}{9} < 0$.

Hence by the Intermediate Value Theorem, there is some c in $(0, \frac{1}{2})$ such that $f(c) = 0$; i.e. a solution to the equation.

Notes:

- Since there is no interval given, you need to choose your own. There are many different intervals which will work BUT the intermediate value theorem does not apply for any interval containing -1, because $f(t)$ will not be continuous on that interval.

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[10] 6. Find $\dfrac{dy}{dx}$ at $(0, \pi/4)$ if x and y satisfy the relation

$$x\,\sin(y) = e^y - e^{\pi/4}.$$

$$\begin{aligned}
\frac{d}{dx}\left(x\,\sin(y)\right) &= \frac{d}{dx}\left(e^y - e^{\pi/4}\right) \\
\Rightarrow \quad x\cos(y)\frac{dy}{dx} + \sin(y) &= e^y\frac{dy}{dx} \\
\Rightarrow \quad x\cos(y)\frac{dy}{dx} - e^y\frac{dy}{dx} &= -\sin(y) \\
\Rightarrow \quad \frac{dy}{dx}\left(x\cos(y) - e^y\right) &= -\sin(y) \\
\Rightarrow \quad \frac{dy}{dx} &= \frac{-\sin(y)}{x\cos(y) - e^y}.
\end{aligned}$$

At $(0, \frac{\pi}{4})$:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{-\frac{\sqrt{2}}{2}}{0 - e^{\frac{\pi}{4}}} \\
&= \frac{\sqrt{2}}{2e^{\frac{\pi}{4}}}.
\end{aligned}$$

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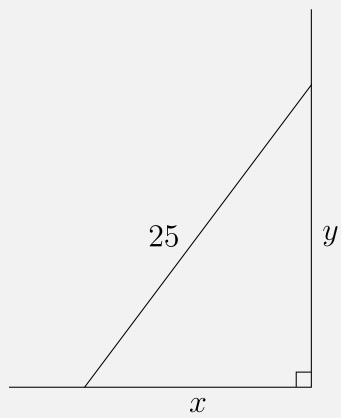
[10] 7. A 25 ft ladder is leaning against a vertical wall. The ladder begins sliding down the wall. When the base of the ladder is 15 ft from the wall, the base's velocity is 2 ft/s. What is the velocity of the top of the ladder at the same time?



Let x be the distance from the base of the ladder to the wall.
Let y be the distance from the top of the ladder to the ground.

We know $\dfrac{dx}{dt} = 2$ when $x = 15$. We want $\dfrac{dy}{dt}$ when $x = 15$.

$$\begin{aligned} x^2 + y^2 &= 25 \\ \Rightarrow \qquad 2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 0. \end{aligned}$$

When $x = 15$, $y = \sqrt{25^2 - 15^2} = \sqrt{400} = 20$. So

$$2(15)(2) + 2(20)\frac{dy}{dt} = 0$$

$$\Rightarrow \qquad \frac{dy}{dt} = -\frac{60}{40} = -\frac{3}{2} \text{ ft/sec.}$$

That is, the velocity of the top of the ladder is $-\frac{3}{2}$ ft/sec when the base of the ladder is 15 ft from the wall.