THE UNIVERSITY OF MANITOBA

October 28, 2016 MIDTERM EXAMINATION

DEPARTMENT & COURSE NO: Math 1500 COVER PAGE

EXAMINATION: Intro. to Calculus TIME: 1 HOUR EXAMINER: Various

NAME: (Print in ink) ____________________

STUDENT NUMBER: ____________________

SIGNATURE: (in ink) ____________________

(I understand that cheating is a serious offense)

❑	A01	10:30-11:20 MWF	T. Kucera
❑	A02	9:30-10:20 MWF	R. Borgersen
❑	A03	11:30-12:20 MWF	K. Gunderson
❑	A04	12:30-1:20 MWF	N. Harland
❑	A05	11:30-12:45 TTh	S. Kalajdzievski
❑	A06	19:00-22:00 T	F. Magpantay
❑	A07	10:00-11:15 TTh	F. Ghahramani
❑	A08	8:30-9:20 MWF	S. Kalajdzievski
❑	Challenge for credit		
❑	Distance Education		

INSTRUCTIONS TO STUDENTS:

This is a 1 hour exam. Show all your work and **justify** your answers. **Unjustified answers will receive LITTLE or NO CREDIT.**

No aids or electronic devices of any kind are permitted during the examination.

This exam has a title page, **6 pages** of questions, including 1 blank page for rough work. Please check that you have all the pages. You may remove the blank pages if you want, but be careful not to loosen the staples.

The value of each question is indicated beside the statement of the question. The total value of all questions is **60 points**.

Question	Points	Score
1	15	
2	8	
3	8	
4	11	
5	9	
6	9	

TOTAL	60	

Answer all questions on the exam paper in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

[15] 1. Calculate each of the following limits if they exist. If the limit does not exist, determine whether the limit is ∞, $-\infty$ or neither.

[3] (a) $\lim\limits_{x\to\infty} e^{\left(\frac{1}{x^2}\right)}$ [Explain your argument in one sentence.]

Solution. $\lim\limits_{x\to\infty} e^{\left(\frac{1}{x^2}\right)} = 1$ since $\lim\limits_{x\to\infty}\frac{1}{x^2} = 0$, and since exponential functions are continuous.

[4] (b) $\lim\limits_{x\to 0^+}\frac{x}{2-\sqrt{x^2+4}}$

Solution. $\lim\limits_{x\to 0^+}\frac{x}{2-\sqrt{x^2+4}} = \lim\limits_{x\to 0^+}\frac{x}{2-\sqrt{x^2+4}}\frac{2+\sqrt{x^2+4}}{2+\sqrt{x^2+4}} = \lim\limits_{x\to 0^+}\frac{x(2+\sqrt{x^2+4})}{4-(x^2+4)} = \lim\limits_{x\to 0^+}\frac{(2+\sqrt{x^2+4})}{-x} = -\infty$; in the last step we note that the numerator tends to 4 while the denominator approaches 0 through negative numbers.

[4] (c) $\lim\limits_{x\to-\infty}\frac{x+\sqrt{4x^2+x}}{x-3}$

Solution.

$$\lim_{x\to-\infty}\frac{x+\sqrt{4x^2+x}}{x-3} = \lim_{x\to-\infty}\frac{x+\sqrt{x^2(4+1/x)}}{x-3} = \lim_{x\to-\infty}\frac{x+(-x)\sqrt{(4+1/x)}}{x(1-3/x)} =$$

$$= \lim_{x\to-\infty}\frac{x\left(1-\sqrt{(4+1/x)}\right)}{x(1-3/x)} = \lim_{x\to-\infty}\frac{1-\sqrt{(4+1/x)}}{(1-3/x)} = -1.$$

[4] (d) Let $f(x)$ be a function defined on an open interval containing 0, and suppose $-1\le f(x)\le 2$. Calculate $\lim\limits_{x\to 0^+}\sqrt{x}\, f(x)$; justify your answer. [Hint: Squeeze theorem!]

Solution. Start with $-1\le f(x)\le 2$ and multiply the inequalities by $\sqrt{x}$; this does not affect the orientation of the inequality since we multiply by a positive number. Get: $-\sqrt{x}\le\sqrt{x}\, f(x)\le 2\sqrt{x}$. Now apply the limit, as x approaches 0 from the positive side, to all three expressions; the limits preserve the (orientation of) inequalities, and so we get $\lim\limits_{x\to 0^+}\left(-\sqrt{x}\right)\le\lim\limits_{x\to 0^+}\left(\sqrt{x}\, f(x)\right)\le\lim\limits_{x\to 0^+}\left(2\sqrt{x}\right)$. Since $\lim\limits_{x\to 0^+}\left(-\sqrt{x}\right)=0$, and since $\lim\limits_{x\to 0^+}\left(2\sqrt{x}\right)=0$, it follows from the Squeeze Theorem that $\lim\limits_{x\to 0^+}\left(\sqrt{x}\, f(x)\right)=0$.

[8] 2. Consider the following function: $f(x)=\begin{cases} x^2 & \text{if } x\le 1 \\ 3x-2 & \text{if } 1<x\le 3 \\ e^{x-3}+a^2+5 & \text{if } x>3 \end{cases}$.

[3] (a) Show that $f(x)$ is continuous at $x=1$.

Solution. We need to confirm that $\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^+} f(x)=f(1)$. We compute:

$\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-} x^2=1$; $\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+}(3x-2)=1$; $f(1)=1^2=1$, and so all three are equal.

[5] (b) Find all real numbers a for which the function $f(x)$ is continuous at $x=3$.

Solution. We want to find all a such that $\lim\limits_{x\to 3^-} f(x)=\lim\limits_{x\to 3^+} f(x)=f(3)$. Computation:

$\lim\limits_{x\to 3^-} f(x)=\lim\limits_{x\to 3^-}(3x-2)=7$; $\lim\limits_{x\to 3^+} f(x)=\lim\limits_{x\to 3^+}(e^{x-3}+a^2+5)=1+a^2+5=a^2+6$; $f(3)=7$. Hence $\lim\limits_{x\to 3^-} f(x)=\lim\limits_{x\to 3^+} f(x)=f(3)$ becomes $7=a^2+6=7$, which after solving yields $a=1$ or $a=-1$.

[8] 3. Prove that if a function $f(x)$ is differentiable at the point where $x=a$, then $f(x)$ is continuous at the point where $x=a$.

Solution. Suppose $f(x)$ is differentiable at the point where $x=a$; this means that $\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$ exists (and is a finite number). We want to show that $f(x)$ is continuous at the point where $x=a$, i.e. that $\lim\limits_{x\to a} f(x)=f(a)$.

Step 1: $\lim\limits_{x\to a}\big(f(x)-f(a)\big)=\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}(x-a)\overset{(1)}{=}\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}\lim\limits_{x\to a}(x-a)\overset{(2)}{=}0$, where in equality (1) we used the assumption that $\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$ exists, and in (2) we merely noticed that $\lim\limits_{x\to a}(x-a)=0$.

Step 2: $\lim\limits_{x\to a} f(x)=\lim\limits_{x\to a}(f(x)-f(a)+f(a))\overset{(1)}{=}\lim\limits_{x\to a}(f(x)-f(a))+\lim\limits_{x\to a} f(a)\overset{(2)}{=}0+f(a)=f(a)$, where in (1) we used the fact that both limits exist (the first one by Step 1), and in (2) we used what we have computed in Step 1.

[11] 4. Find $f'(x)$. Do NOT simplify your answer after you evaluate the derivative.

[3] (a) $f(x) = e^3 + e^{2x} + \dfrac{1}{\sqrt[5]{x^2}}$

Solution.

$$f'(x) = 0 + 2e^{2x} + \left(-\frac{2}{5}\right)x^{-\frac{7}{5}}.$$

[3] (b) $f(x) = (1-10x^3)^{12}$

Solution.

$$f'(x) = 12\,(1-10x^3)^{11}(-30x^2).$$

[5] (c) $f(x) = (\sin x)\dfrac{1-\sqrt[3]{x^2}}{1+x}$

Solution.

$$f'(x) = (\cos x)\frac{1-\sqrt[3]{x^2}}{1+x} + (\sin x)\frac{\left(-\frac{2}{3}\right)x^{\left(-\frac{1}{3}\right)}(1+x) - \left(1-\sqrt[3]{x^2}\right)}{(1+x)^2}$$

[9] 5. [2] (a) State the definition of the derivative of a function $f(x)$.

[5] (b) Use the definition of the derivative to evaluate $f'(x)$ if $f(x)=x^2+2x+3$. (No points will be awarded if other methods are used.)

[2] (c) Find the equation in the form $y=mx+b$ of the tangent line to $f(x)=x^2+2x+3$ at the point where $x=1$.

Solution

(a) $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$.

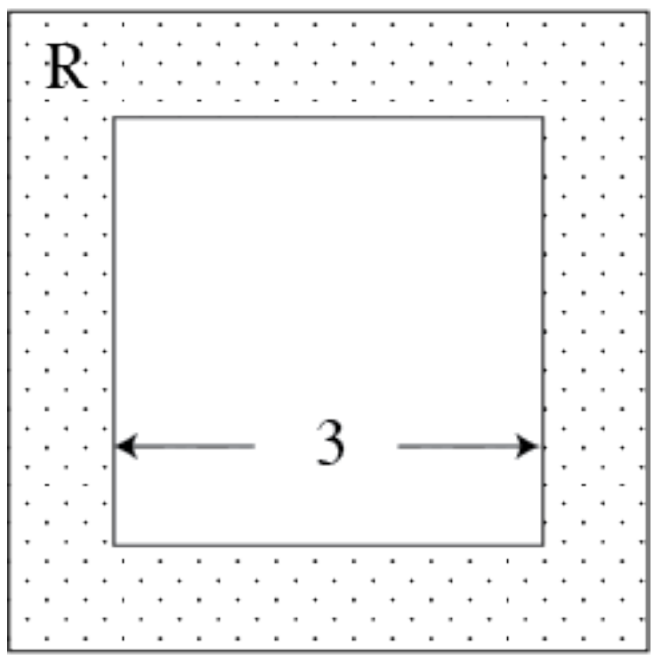
(b)

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^2+2(x+h)+3-(x^2+2x+3)}{h}=\lim_{h\to 0}\frac{x^2+2xh+h^2+2x+2h+3-x^2-2x-3}{h}=$$

$$=\lim_{h\to 0}\frac{2xh+h^2+2h}{h}=\lim_{h\to 0}\frac{h(2x+h+2)}{h}=\lim_{h\to 0}(2x+h+2)=2x+2$$

(c) It follows from part (b) that $f'(1)=4$. Hence the equation of the desired tangent line is $y=4x+b$. Notice that when $x=1$ then $f(x)=6$. Hence the tangent line passes through $(1,6)$, which means that the coordinates of this point satisfy the equation of the tangent line: $6=(4)(1)+b$, so that $b=2$. Hence the tangent line that we wanted is $y=4x+2$.

[9] 6. Consider the region R bounded between two squares, as depicted in the illustration: the smaller square has a side of fixed length equal to 3 metres, while the larger square is increasing. If the area of R is changing at the rate of 20 m^2/sec, what is the rate of increase of the side-length of the larger square at the moment when this side-length is 6 metres?



Solution. Denote the edge-length of the larger square by x, and denote the area of R by A. We find easily that $A = x^2 - 9$. We are given that $\frac{dA}{dt} = 20\ m^2/\text{sec}$; we want to find $\frac{dx}{dt}$ at the moment when $x = 6\ m$.

We differentiate $A = x^2 - 9$ with respect to time t: $\frac{dA}{dt} = 2x\frac{dx}{dt}$. At the given moment we get $20 = (2)(6)\frac{dx}{dt}$.

Hence $\frac{dx}{dt} = \frac{10}{6}\ m/\text{sec}$.