Math 1500 Midterm Exam **Solutions** **Fall 2015**

1) Determine the following limits, if they exist. If a limit does not exist, state this fact explicitly and then briefly explain why the limit does not exist.

a) $\lim_{t\to3}\frac{1-\sqrt{t-2}}{t-3}$

$$\lim_{t\to3}\frac{1-\sqrt{t-2}}{t-3}=\lim_{t\to3}\frac{1-\sqrt{t-2}}{t-3}\bullet\frac{1+\sqrt{t-2}}{1+\sqrt{t-2}}=\lim_{t\to3}\frac{1-(t-2)}{(t-3)(1+\sqrt{t-2})}=\lim_{t\to3}\frac{3-t}{(t-3)(1+\sqrt{t-2})}$$

$$=\lim_{t\to3}\frac{-1}{1+\sqrt{t-2}}=\frac{-1}{1+\sqrt{3-2}}=-\frac{1}{2}$$

b) $\lim_{x\to-\infty}\frac{4x-1}{\sqrt{x^2+2x-3}}$

$$\lim_{x\to-\infty}\frac{4x-1}{\sqrt{x^2+2x-3}}=\lim_{x\to-\infty}\frac{\frac{4x-1}{x}}{\frac{\sqrt{x^2+2x-3}}{x}}=\lim_{x\to-\infty}\frac{\frac{4x-1}{x}}{\frac{\sqrt{x^2+2x-3}}{-\sqrt{x^2}}}=\lim_{x\to-\infty}\frac{\frac{4x-1}{x}}{-\sqrt{\frac{x^2+2x-3}{x^2}}}$$

$$=\lim_{x\to-\infty}\frac{4-\frac{1}{x}}{-\sqrt{1+\frac{2}{x}-\frac{3}{x^2}}}=\frac{4-0}{-\sqrt{1+0-0}}=-4$$

c) $\lim_{x\to2^+}\left(\sqrt{x-2}\,\cos\frac{1}{x-2}\right)$

$-1\le\cos\theta\le1\ \ for\ all\ \theta.$ Therefore: $-1\le\cos\frac{1}{x-2}\le1$

Therefore: $-1\sqrt{x-2}\le\sqrt{x-2}\cos\frac{1}{x-2}\le1\sqrt{x-2}\implies-\sqrt{x-2}\le\sqrt{x-2}\cos\frac{1}{x-2}\le\sqrt{x-2}$

Since $\lim_{x\to2^+}\left(-\sqrt{x-2}\right)=-\sqrt{2-2}=0$ and $\lim_{x\to2^+}\left(\sqrt{x-2}\right)=\sqrt{2-2}=0,$

therefore, by the squeeze rule, $\lim_{x\to2^+}\left(\sqrt{x-2}\,\cos\frac{1}{x-2}\right)=0$

2) State the conditions required for continuity of a function at a point and, by using these conditions, determine the values of a and c that will make the following piecewise-defined function continuous at $x=2$.

$$f(x)=\begin{cases}cx^2 & if\ x<2\\ a & if\ x=2\\ cx+5 & if\ x>2\end{cases}$$

For continuity at $x=2$, $\lim_{x\to2}f(x)$ must exist.
For this limit to exist, we require that $\lim_{x\to2^-}f(x)=\lim_{x\to2^+}f(x)$.

$\lim_{x\to2^-}f(x)=\lim_{x\to2^-}(cx^2)=c(2)^2=4c$
$\lim_{x\to2^+}f(x)=\lim_{x\to2^+}(cx+5)=c(2)+5=2c+5$
Therefore, $4c=2c+5\ \Rightarrow\ c=\frac{5}{2}$.
With this value of c, $\lim_{x\to2}f(x)=4c\ (or\ 2c+5)=4\left(\frac{5}{2}\right)=10$.

For continuity at $x=2$, $f(2)$ must be defined. It is: $f(2)=a$.

For continuity at $x = 2$, we also require $\lim_{x\to 2} f(x) = f(2)$.
As $f(2) = a$ and $\lim_{x\to 2} f(x) = 10$, we have $10 = a \;\; \Rightarrow a = 10$!
For the function f to be continuous at $x = 2$, $a = 10 \; and \; c = \frac{5}{2}$.

3) Given that $f(x) = \sin(x)$. By using the definition of derivative, prove that $f'(x) = \cos(x)$.
You may use the following two limits without justification:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \text{ and } \lim_{x\to 0} \frac{\cos(x)-1}{x} = 0$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} = \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h - \sin x + \cos x \sin h}{h} = \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h\to 0} \frac{\sin x(\cos h-1)}{h} + \lim_{h\to 0} \frac{\cos x \sin h}{h} = \lim_{h\to 0} \sin x \frac{(\cos h - 1)}{h} + \lim_{h\to 0} \cos x \frac{\sin h}{h}$$

$$= \lim_{h\to 0} \sin x \bullet \lim_{h\to 0} \frac{(\cos h-1)}{h} + \lim_{h\to 0} \cos x \bullet \lim_{h\to 0} \frac{\sin h}{h}$$

$$= \sin x \bullet 0 + \cos x \bullet 1 = \cos x$$

4) Find the requested derivative for each of the following functions; do <u>not</u> simplify your answers.

a) Let $f(x) = e^x tanx$. Find $f'(x)$.

$$\frac{df(x)}{dx} = \tan(x) \frac{de^x}{dx} + e^x \frac{d\tan(x)}{dx} = \tan(x)\, e^x + e^x \sec^2(x)$$

b) Let $g(x) = \left(1 + \sqrt{x}\right)^5$. Find $g'(x)$.

$$\frac{dg(x)}{dx} = 5\left(1+\sqrt{x}\right)^4 \frac{d(1+\sqrt{x})}{dx} = 5\left(1+\sqrt{x}\right)^4 \left(0 + \frac{1}{2}x^{-\frac{1}{2}}\right)$$

c) Let $v(t) = \frac{1+t^3}{1+e^{2t}}$. Find $v'(t)$.

$$\frac{dv(t)}{dt} = \frac{(1+e^{2t})\frac{d(1+t^3)}{dx} - (1+t^3)\frac{d(1+e^{2t})}{dx}}{(1+e^{2t})^2} = \frac{(1+e^{2t})(0+3t^2) - (1+t^3)\left(0+e^{2t}\frac{d(2t)}{dx}\right)}{(1+e^{2t})^2}$$

$$= \frac{(1+e^{2t})(0+3t^2) - (1+t^3)\left(0+e^{2t}(2)\right)}{(1+e^{2t})^2}$$

5) Given the relation defined by the equation $y = xe^y + 1$.
Determine the equation of the tangent line to the curve at the point $(-1,0)$ on the curve.

$$\frac{d(y)}{dx} = \frac{d(xe^y+1)}{dx} \Rightarrow y' = [(1)e^y + xe^y y'] + 0 \Rightarrow y' = e^y + xe^y y'$$

$$\Rightarrow y' - xe^y y' = e^y \Rightarrow (1 - xe^y)y' = e^y \Rightarrow y' = \frac{e^y}{1-xe^y}$$

At the point $(-1,0)$, $y' = \frac{e^0}{1-(-1)e^0} = \frac{1}{2}$

Therefore, the equation of the tangent line to the curve at this point is: $y - 0 = \frac{1}{2}(x + 1)$

6) Sand is poured in a conical pile at a rate of 20 m^3/minute. The diameter of the cone is always equal to its height. How fast is the height of the conical pile increasing when the pile is 10 m high?
(The volume V, the radius r and the height h of a circular cone are related by the formula $V = \frac{1}{3}\pi r^2 h$)

Since $h = 2r \;\Rightarrow\; r = \frac{h}{2}$, then $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3$.
Differentiating with respect to time, we get: $\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$.
It is given that $\frac{dy}{dx} = 20\, m^3/minute$ and we need to find $\frac{dh}{dt}$ when $h = 10\, m$.
Thus, $20 = \frac{1}{4}\pi(10)^2 \frac{dh}{dt} \;\Rightarrow\; \frac{dh}{dt} = \frac{20}{25\pi} = \frac{4}{5\pi}$ and so the height of the pile of sand is increasing at a rate of $\frac{4}{5\pi}\, m/min$ when the pile is 10 m high.