Math 1500 Midterm Exam **Solutions** **Fall 2014**

1)a) $\lim_{x\to 0}\frac{\sqrt{x^2+1}-1}{x} = \lim_{x\to 0}\frac{\sqrt{x^2+1}-1}{x} \bullet \frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}+1} = \lim_{x\to 0}\frac{(x^2+1)-1}{x(\sqrt{x^2+1}+1)} = \lim_{x\to 0}\frac{x^2}{x(\sqrt{x^2+1}+1)} =$

$\lim_{x\to 0}\frac{x}{\sqrt{x^2+1}+1} = \frac{0}{\sqrt{0^2+1}+1} = 0$

b) $\lim_{x\to -1^+}\frac{2x-3}{x^2-1} = \lim_{x\to -1^+}\frac{2x-3}{(x+1)(x-1)} = \frac{2(-1)-3}{(-1^+ +1)(-1-1)} = \frac{-5}{(0^+)(-2)} = \frac{-5}{0^-} \to +\infty$ Limit $\nexists$.

Or

Upon substitution, one gets the form $\frac{-5}{0}$ so the limit does not exist but the functional values tend to $\pm\infty$. The question is: which one?!

The numerator approaches a negative number as $x \to -1$.

The denominator factors as $(x+1)(x-1)$ and the first factor approaches 0 from the positive side as $x \to -1$ and the second factor approaches a negative number as $x \to -1$, which makes the full denominator approaching 0 from the negative side as $x \to -1$.

As the numerator is approaching a negative number as $x \to -1$ and the denominator is approaching 0 from the negative side as $x \to -1$, we have determined that the functional values are overall positive as $x \to -1$.

From the first observation, we conclude that the functional values are increasing without bound.

(aka: $+\infty$)

c) $\lim_{x\to -\infty}\frac{\sqrt{2x^2+1}}{x+2} = \lim_{x\to -\infty}\frac{\sqrt{x^2\left(2+\frac{1}{x^2}\right)}}{x\left(1+\frac{2}{x}\right)} = \lim_{x\to -\infty}\frac{\sqrt{x^2}\sqrt{2+\frac{1}{x^2}}}{x\left(1+\frac{2}{x}\right)} = \lim_{x\to -\infty}\frac{-x\sqrt{2+\frac{1}{x^2}}}{x\left(1+\frac{2}{x}\right)} =$

$\lim_{x\to -\infty}\frac{-\sqrt{2+\frac{1}{x^2}}}{1+\frac{2}{x}} = \frac{-\sqrt{2+0}}{1+0} = -\sqrt{2}$

Or

$\lim_{x\to -\infty}\frac{\sqrt{2x^2+1}}{x+2} = \lim_{x\to -\infty}\frac{\frac{\sqrt{2x^2+1}}{x}}{\frac{x+2}{x}} = \lim_{x\to -\infty}\frac{\frac{\sqrt{2x^2+1}}{-\sqrt{x^2}}}{\frac{x+2}{x}} = \lim_{x\to -\infty}\frac{-\sqrt{\frac{2x^2+1}{x^2}}}{\frac{x+2}{x}} = \lim_{x\to -\infty}\frac{-\sqrt{2+\frac{1}{x^2}}}{1+\frac{2}{x}} =$

$\frac{-\sqrt{2+0}}{1+0} = -\sqrt{2}$

d) $\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\frac{\frac{\sin x}{\cos x}}{x}=\lim_{x\to 0}\frac{\sin x}{x\cos x}=\lim_{x\to 0}\frac{\sin x}{x}\bullet\lim_{x\to 0}\frac{1}{\cos x}=1\bullet\frac{1}{\cos 0}=\frac{1}{1}=1$

2)a) $f(x)=(4-x^2)e^x \;\rightarrow\; f'(x)=(4-x^2)'e^x+(4-x^2)(e^x)'=(-2x)e^x+(4-x^2)e^x$

b) $f(x)=\frac{x^2+1}{\cos x+1} \;\rightarrow\; f'(x)=\frac{(\cos x+1)(x^2+1)'-(x^2+1)(\cos x+1)'}{(\cos x+1)^2}=\frac{(\cos x+1)(2x)-(x^2+1)(-\sin x)}{(\cos x+1)^2}$

c) $f(x)=\tan(1+\sin x) \;\rightarrow\; f'(x)=\sec^2(1+\sin x)\bullet(\cos x)$

3) $f(x)=\begin{cases}x^2 & x<1\\ 2 & x=1\\ e^{x-1} & x>1\end{cases}$

a) $\lim_{x\to 1^+}f(x)=\lim_{x\to 1^+}e^{x-1}=e^{1-1}=e^0=1$

$\lim_{x\to 1^-}f(x)=\lim_{x\to 1^+}x^2=1^2=1$

b) Is f continuous at $x=1$?
Condition #1: Does $\lim_{x\to 1}f(x)$ exist? (ie. does $\lim_{x\to 1^-}f(x)=\lim_{x\to 1^+}f(x)$?)
Yes. The two one-sided limits both equal 1.
Condition #2: Is $f(1)$ defined?
Yes it is. $f(1)=2$
Condition #3: Does $\lim_{x\to 1}f(x)=f(1)$?
No it does not. $1\neq 2$
Therefore, as all three conditions for continuity are <u>not</u> met, f is <u>not</u> continuous at $x=1$.

4) $f(x)=\frac{1}{x}$

$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}=\lim_{h\to 0}\frac{\frac{1}{x+h}\left(\frac{x}{x}\right)-\frac{1}{x}\left(\frac{x+h}{x+h}\right)}{h}=\lim_{h\to 0}\frac{\frac{x-(x+h)}{x(x+h)}}{h}=$

$\lim_{h\to 0}\frac{\frac{-h}{x(x+h)}}{h}=\lim_{h\to 0}\frac{-h}{xh(x+h)}=\lim_{h\to 0}\frac{-1}{x(x+h)}=\frac{-1}{x(x+0)}=\frac{-1}{x^2}$

5) Given: $g(x) = f(x^2 + 1)$ and $f'(2) = -1$. Find: $g'(1)$.

Using the chain rule for a composite function: $g'(x) = f'(x^2 + 1)\bullet(x^2 + 1)' = f'(x^2 + 1)\bullet 2x$

With substitution, $g'(1) = f'(1^2 + 1)\bullet 2(1) = f'(2)\bullet 2 = -1\bullet 2 = -2$

6) Prove: if $f'(x)$ and $g'(x)$ exist, then $(f(x) + g(x))' = f'(x) + g'(x)$

$(f(x) + g(x))' = \lim_{h\to 0} \frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} = \lim_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} =$

$\lim_{h\to 0} \frac{f(x+h)-f(x)+g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{[f(x+h)-f(x)]+[g(x+h)-g(x)]}{h} =$

$\lim_{h\to 0} \left[\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right] = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = f'(x) + g'(x).$

7) Determine an equation of the tangent line to the curve $y^2 - xy + x^2 = 1$ at the point $(1,1)$.

By using implicit differentiation:

$\frac{d(y^2-xy+x^2)}{dx} = \frac{d(1)}{dx} \rightarrow 2y\bullet y' - [(1)y + xy') + 2x = 0 \rightarrow 2yy' - y - xy' + 2x = 0 \rightarrow$

$2yy' - xy' = y - 2x \rightarrow (2y - x)y' = y - 2x \rightarrow y' = \frac{y-2x}{2y-x}$

At the point $(1,1)$, $y' = \frac{(1)-2(1)}{2(1)-(1)} = -1$.

Using the formula $y - y_1 = m(x - x_1)$, we obtain $y - 1 = -1(x - 1)$ as the tangent line.

<u>Or</u>

$\frac{d(y^2-xy+x^2)}{dx} = \frac{d(1)}{dx} \rightarrow 2y\bullet y' - [(1)y + xy')] + 2x = 0$

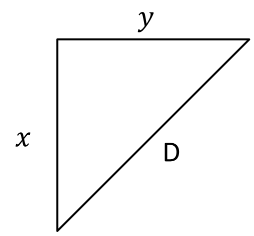
Upon substitution of the point $(1,1)$, we get $2(1)\bullet y' - [1(1) + (1)y')] + 2(1) = 0$

which yields $2y' - 1 - y' + 2 = 0 \rightarrow y' = 1$.

Using the formula $y - y_1 = m(x - x_1)$, we obtain $y - 1 = -1(x - 1)$ as the tangent line.

8) Yertle the turtle(s) ...

Given: $\frac{dy}{dt} = -20\ m/h$ and $\frac{dx}{dt} = -30\ m/h$



West and North are at right angles to each other.

Goal: Find $\frac{dD}{dt}$ when $x = 3\ m$ and $y = 4\ m$.

Relationship: $D^2 = x^2 + y^2$

As $D, x\ and\ y$ are all implicit functions of time, we can perform implicit differentiation wrt time.

$\frac{d(D^2)}{dt} = \frac{d(x^2+y^2)}{dx} \rightarrow 2D\bullet\frac{dD}{dt} = 2x\bullet\frac{dy}{dt} + 2y\frac{dy}{dt} \rightarrow D\bullet\frac{dD}{dt} = x\bullet\frac{dy}{dt} + y\frac{dy}{dt}$

This yields $\frac{dD}{dt} = \frac{x\bullet\frac{dy}{dt}+y\frac{dy}{dt}}{D}$.

Aside: Find D when $x = 3\ m$ and $y = 4\ m$. $D^2 = x^2 + y^2 \rightarrow D^2 = 3^2 + 4^2 \rightarrow D = 5$

Therefore: $\frac{dD}{dt} = \frac{x\bullet\frac{dy}{dt}+y\frac{dy}{dt}}{D} = \frac{3\bullet(-30)+4(-20)}{5} = \frac{-170}{5} = -34$

The distance between the turtles is decreasing at a rate of $34\ m/h$ at this time.