136.271
Midterm Exam 2 Solutions
November 21, 2001

(50 minutes; **justify** your answers unless otherwise stated; no calculators)

1. Find the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n3^n}$.

Using the ration test: $\lim_{n\to\infty}\left|\frac{\frac{(3x-2)^{n+1}}{(n+1)3^{n+1}}}{\frac{(3x-2)^n}{n3^n}}\right| = |3x-2|\lim_{n\to\infty}\frac{n}{3(n+1)} = \frac{1}{3}|3x-2|$. So, the series converges if $\frac{1}{3}|3x-2|<1$, i.e. if $|3x-2|<3$, i.e. if $-3<3x-2<3$, i.e. if $-1<3x<5$, which finally tells us that $-\frac{1}{3}<x<\frac{5}{3}$.

We now consider the edges: $x=-\frac{1}{3}$ yields the series $\sum_{n=1}^{\infty}\frac{(-1-2)^n}{n3^n}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, which converges by the alternating series test. On the other hand for $x=\frac{5}{3}$ we get the series $\sum_{n=1}^{\infty}\frac{(3)^n}{n3^n}=\sum_{n=1}^{\infty}\frac{1}{n}$ which diverges. So, the interval of convergence is $\left[-\frac{1}{3},\frac{5}{3}\right)$.

2. Find the sum of each of the series $\sum_{n=2}^{\infty} n(n-1)x^n$ and $\sum_{n=2}^{\infty}\frac{n(n-1)}{2^n}$.

$$\sum_{n=2}^{\infty} n(n-1)x^n = x^2\sum_{n=2}^{\infty} n(n-1)x^{n-2} = x^2\sum_{n=2}^{\infty}\left(x^n\right)'' \overset{\text{in the interval of convergence}}{=} x^2\left(\sum_{n=2}^{\infty}x^n\right)'' =$$

$$= x^2\left(\sum_{n=0}^{\infty}x^n - 1 - x\right)'' = x^2\left(\frac{1}{1-x}-1-x\right)'' = x^2\,\frac{2}{(1-x)^3}$$

(All this could be done in the interval of convergence of the series $\sum_{n=0}^{\infty}x^n$, which is $-1<x<1$).

Now note that $\sum_{n=2}^{\infty}\frac{n(n-1)}{2^n}$ is what we get from $\sum_{n=2}^{\infty}n(n-1)x^n$ when $x=\frac{1}{2}$. Consequently

$$\sum_{n=2}^{\infty}\frac{n(n-1)}{2^n} = \left(\frac{1}{2}\right)^2\frac{2}{\left(1-\frac{1}{2}\right)^3} = 4$$

3. Find the sum of the series $\sum_{n=0}^{\infty}\frac{3^n}{5^{n+1}n!}$. [Hint: use a known Maclaurin representation.]

$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ for all x. Put $x=\frac{3}{5}$ to get $e^{3/5}=\sum_{n=0}^{\infty}\frac{3^n}{5^n n!} = 5\sum_{n=0}^{\infty}\frac{3^n}{5^{n+1}n!}$.

Consequently $\sum_{n=0}^{\infty}\frac{3^n}{5^{n+1}n!}=\frac{1}{5}e^{3/5}$.

4. Evaluate the integral $\int_0^x \frac{1}{1+t^4}\,dt$ as a power series. Find the interval of convergence of that power series.

First of all we have $\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n$ for $-1<x<1$. So,
$\frac{1}{1+t^4}=\frac{1}{1-(-t^4)}=\sum_{n=0}^{\infty}(-t^4)^n=\sum_{n=0}^{\infty}(-1)^n t^{4n}$. This is fine for $-1<-t^4<1$, i.e. for $-1<t<1$. For such values of t and for $-1<x<1$ we can integrate term by term to get $\int_0^x \frac{1}{1+t^4}\,dt=\int_0^x \sum_{n=0}^{\infty}(-1)^n t^{4n}\,dt=\sum_{n=0}^{\infty}(-1)^n \frac{t^{4n+1}}{4n+1}\Big|_0^x=\sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+1}}{4n+1}$.

5. Use the binomial formula to expand the function $\frac{x}{\sqrt{4+x^2}}$ as a power series. You do **NOT** need to simplify your answer. What is the **radius** of convergence of the power series.

First of all $\frac{1}{\sqrt{4+x^2}}=\left(4+x^2\right)^{-1/2}=\frac{1}{2}\left(1+\left(\frac{x}{2}\right)^2\right)^{-1/2}$. By the binomial formula $\frac{1}{2}\left(1+\left(\frac{x}{2}\right)^2\right)^{-1/2}=\frac{1}{2}\left(1+\sum_{k=1}^{\infty}\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\ldots\left(-\frac{1}{2}+k-1\right)}{k!}\left(\left(\frac{x}{2}\right)^2\right)^k\right)$. As a consequence we have that

$$\frac{x}{\sqrt{4+x^2}}=\frac{1}{2}x\left(1+\left(\frac{x}{2}\right)^2\right)^{-1/2}=\frac{1}{2}\left(x+\sum_{k=1}^{\infty}\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\ldots\left(-\frac{1}{2}+k-1\right)}{k!}\left(\frac{x}{2}\right)^{2k+1}\right).$$

This is true for $-1<\left(\frac{x}{2}\right)^2<1$, which means $-2<x<2$. So, the radius of convergence is 2.