136.271
Midterm Exam 2 Solutions
November 21, 2001

(50 minutes; **justify** your answers unless otherwise stated; no calculators)

1. Find the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n3^n}$.

Using the ration test: $\lim_{n\to\infty}\left|\dfrac{\frac{(3x-2)^{n+1}}{(n+1)3^{n+1}}}{\frac{(3x-2)^n}{n3^n}}\right| = |3x-2| \lim_{n\to\infty} \frac{n}{3(n+1)} = \frac{1}{3}|3x-2|$. So, the series

converges if $\frac{1}{3}|3x-2| < 1$, i.e. if $|3x-2| < 3$, i.e. if $-3 < 3x-2 < 3$, i.e. if $-1 < 3x < 5$,

which finally tells us that $-\frac{1}{3} < x < \frac{5}{3}$.

We now consider the edges: $x = -\frac{1}{3}$ yields the series $\sum_{n=1}^{\infty} \frac{(-1-2)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which converges by the alternating series test. On the other hand for $x = \frac{5}{3}$ we get the series $\sum_{n=1}^{\infty} \frac{(3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges. So, the interval of convergence is $\left[-\frac{1}{3}, \frac{5}{3}\right)$.

2. Find the sum of each of the series $\sum_{n=2}^{\infty} n(n-1)x^n$ and $\sum_{n=2}^{\infty} \frac{n(n-1)}{2^n}$.

$$\sum_{n=2}^{\infty} n(n-1)x^n = x^2 \sum_{n=2}^{\infty} n(n-1)x^{n-2} = x^2 \sum_{n=2}^{\infty} \left(x^n\right)'' \overset{\text{in the interval of convergence}}{=} x^2 \left(\sum_{n=2}^{\infty} x^n\right)'' =$$

$$= x^2\left(\sum_{n=0}^{\infty} x^n - 1 - x\right)'' = x^2\left(\frac{1}{1-x} - 1 - x\right)'' = x^2\left(\frac{2}{(1-x)^3}\right)$$

(All this could be done in the interval of convergence of the series $\sum_{n=0}^{\infty} x^n$, which is $-1 < x < 1$).

Now note that $\sum_{n=2}^{\infty} \frac{n(n-1)}{2^n}$ is what we get from $\sum_{n=2}^{\infty} n(n-1)x^n$ when $x = \frac{1}{2}$. Consequently

$$\sum_{n=2}^{\infty} \frac{n(n-1)}{2^n} = \left(\frac{1}{2}\right)^2 \frac{2}{\left(1-\frac{1}{2}\right)^3} = 4$$

3. Find the sum of the series $\sum_{n=0}^{\infty} \frac{3^n}{5^{n+1}\, n!}$. [Hint: use a known Maclaurin representation.]

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x. Put $x = \frac{3}{5}$ to get $e^{3/5} = \sum_{n=0}^{\infty} \frac{3^n}{5^n\, n!} = 5\sum_{n=0}^{\infty} \frac{3^n}{5^{n+1}\, n!}$.

Consequently $\sum_{n=0}^{\infty} \frac{3^n}{5^{n+1}\, n!} = \frac{1}{5} e^{3/5}$.

4. Evaluate the integral $\int_0^x \frac{1}{1+t^4}\,dt$ as a power series. Find the interval of convergence of that power series.

First of all we have $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $-1 < x < 1$. So,
$\frac{1}{1+t^4} = \frac{1}{1-(-t^4)} = \sum_{n=0}^{\infty}(-t^4)^n = \sum_{n=0}^{\infty}(-1)^n t^{4n}$. This is fine for $-1 < -t^4 < 1$, i.e. for $-1 < t < 1$. For such values of t and for $-1 < x < 1$ we can integrate term by term to get $\int_0^x \frac{1}{1+t^4}\,dt = \int_0^x \left(\sum_{n=0}^{\infty}(-1)^n t^{4n}\right)dt = \sum_{n=0}^{\infty}(-1)^n \frac{t^{4n+1}}{4n+1}\Big|_0^x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+1}}{4n+1}$.

5. Use the binomial formula to expand the function $\frac{x}{\sqrt{4+x^2}}$ as a power series. You do **NOT** need to simplify your answer. What is the **radius** of convergence of the power series.

First of all $\frac{1}{\sqrt{4+x^2}} = \left(4+x^2\right)^{-1/2} = \frac{1}{2}\left(1+\left(\frac{x}{2}\right)^2\right)^{-1/2}$. By the binomial formula
$2\left(1+\left(\frac{x}{2}\right)^2\right)^{1/2} = \frac{1}{2}\left[1+\sum_{k=1}^{\infty}\frac{-\frac{1}{2}(-\frac{1}{2}-1)\dots(-\frac{1}{2}-k+1)}{k!}\left(\left(\frac{x}{2}\right)^2\right)^k\right]$. As a consequence we have that

$$\frac{x}{\sqrt{4+x^2}} = \frac{1}{2}x\left(1+\left(\frac{x}{2}\right)^2\right)^{-1/2} = \frac{1}{2}\left[x+\sum_{k=1}^{\infty}\frac{-\frac{1}{2}(-\frac{1}{2}-1)\dots(-\frac{1}{2}-k+1)}{k!}\left(\frac{x}{2}\right)^{2k+1}\right].$$

This is true for $-1 < \left(\frac{x}{2}\right)^2 < 1$, which means $-2 < x < 2$. So, the radius of convergence is 2.