Values

[3] 6. [Bonus question]. Suppose that $f(x)$ is a function that satisfies the equation

$$f(x+y) = f(x) + f(y) + x^2y + xy^2$$

for all real numbers x and y. Suppose also that $\lim_{x \to 0} \frac{f(x)}{x} = 1$.

(a) Find $f(0)$. (Hint: Let $x = y = 0$ in the equation.)

$$f(0+0) = f(0) + f(0) + 0 + 0$$

so $f(0) = 2f(0)$, so $f(0) = 0$

(b) Find $f'(0)$. (Hint: Use the definition of derivative.)

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} =$$

$$= \lim_{h \to 0} \frac{f(h)}{h} = 1 \quad \text{(GIVEN !)}$$

(c) Find $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$

$$= \lim_{h \to 0} \frac{\overset{=0}{f(x)} + f(h) + x^2h + xh^2 \overset{=0}{- f(x)}}{h} =$$

$$= \lim_{h \to 0} \frac{f(h) + x^2h + xh^2}{h} = \lim_{h \to 0} \left(\frac{f(h)}{h} + x^2 + xh \right) =$$

$$= 1 + x^2 \;//$$