Midterm exam, 136.130, October 2004: Brief Solutions

(10) 1. Solve, by Gauss-Jordan elimination, the system:

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 7, \\ x_1 + x_2 - x_3 &= 0, \\ 3x_1 + x_2 + 3x_3 &= 14. \end{aligned}$$

Solution. The augmented matrix of the system is $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 1 & 1 & -1 & 0 \\ 3 & 1 & 3 & 14 \end{array}\right]$. The RREF of that matrix is

$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Getting from here to a system, using $x_3 = t$ and solving yields

$x_1 = 7 - 2t,\ x_2 = -7 + 3t,\ x_3 = t$.

(7) 2. Evaluate $\det\begin{bmatrix} 2 & 0 & 11 \\ 1 & 2 & 3 \\ -3 & -2 & -8 \end{bmatrix}$ by row reduction to the determinant of an upper triangular matrix. No other method will be awarded marks. Show all your work.

Solution.

$$A = \begin{bmatrix} 2 & 0 & 11 \\ 1 & 2 & 3 \\ -3 & -2 & -8 \end{bmatrix} \underset{\to}{R_1 \leftrightarrow R_2} A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 11 \\ -3 & -2 & -8 \end{bmatrix} \begin{matrix} (-2)R_1 \text{ to } R_2 \\ (3)R_1 \text{ to } R_3 \\ \to \end{matrix} A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 5 \\ 0 & 4 & 1 \end{bmatrix}$$

$$\underset{\to}{R_2 \text{ to } R_3} A_3 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

We see that $\det A = -\det A_3$ and since $\det A_3 = -24$ it follows that $\det A = 24$.

(10) 3. Evaluate $\det\begin{bmatrix} 5 & 2 & 6 \\ 7 & 3 & 0 \\ 1 & 4 & 8 \end{bmatrix}$ by expansion using column 2. No other method will be awarded marks. Show all your work.

$$\det\begin{bmatrix} 5 & 2 & 6 \\ 7 & 3 & 0 \\ 1 & 4 & 8 \end{bmatrix} = -2\det\begin{bmatrix} 7 & 0 \\ 1 & 8 \end{bmatrix} + 3\det\begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix} - 4\det\begin{bmatrix} 5 & 6 \\ 7 & 0 \end{bmatrix} = 158.$$

---

(9) 4. Let $A=\begin{bmatrix}-40 & 16 & 9\\ 13 & -5 & -3\\ 5 & -2 & -1\end{bmatrix}$

a) Find A^{-1}. Note: A^{-1} has only integer entries.

b) Use A^{-1} to solve $AX=\begin{bmatrix}1\\-2\\-1\end{bmatrix}$.

Solution. (a) Any of the methods we know would give us $A=\begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 3\\ 1 & 0 & 8\end{bmatrix}$.

(b) $X=A^{-1}\begin{bmatrix}1\\-2\\-1\end{bmatrix}=\begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 3\\ 1 & 0 & 8\end{bmatrix}\begin{bmatrix}1\\-2\\-1\end{bmatrix}=\begin{bmatrix}-6\\-11\\-7\end{bmatrix}$.

(10) **5**. Let $A=\begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix}$, $B=\begin{bmatrix}3 & -1 & 2\\ 1 & 0 & 4\end{bmatrix}$, $C=\begin{bmatrix}1 & -1\\ 2 & 1\\ -2 & 3\end{bmatrix}$, $D=\begin{bmatrix}1 & 2 & -1\\ 1 & 0 & 2\\ -2 & -3 & 1\end{bmatrix}$.

Calculate defined expressions; write "undefined" beside undefined expressions.

a) $CA+DB^T=\begin{bmatrix}-3 & -1\\ 12 & 13\\ 6 & -2\end{bmatrix}$

b) $A^2=\begin{bmatrix}7 & 2\\ 3 & 6\end{bmatrix}$

c) $B^2=$ does not exist

d) $AB-3C=$ does not exist

e) $B^TB=\begin{bmatrix}10 & -3 & 10\\ -3 & 1 & -2\\ 10 & -2 & 20\end{bmatrix}$

---

(14) 6. Let $A=\begin{bmatrix}0 & 1\\ 2 & 4\end{bmatrix}$.

a) Using only elementary row operations, transform A into I_2. Use suitable notation (or words) to explain what each elementary row operation is.

$$A=\begin{bmatrix}0 & 1\\ 2 & 4\end{bmatrix}\underset{\rightarrow}{R_1 \leftrightarrow R_2}\begin{bmatrix}2 & 4\\ 0 & 1\end{bmatrix}\underset{\rightarrow}{(1/2)R_1}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\underset{\rightarrow}{(-2)R_2 \; to \; R_1}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

b) Find elementary matrices $E_1, E_2, ..., E_n$, such that $E_n ... E_2 E_1 A = I_2$.

We do to the identity matrix each of the above elementary row operations to get $E_1=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$,

$E_2=\begin{bmatrix}1/2 & 0\\ 0 & 1\end{bmatrix}$, $E_3=\begin{bmatrix}1 & -2\\ 0 & 1\end{bmatrix}$.

c) For each E_i found in b), give the inverse E_i^{-1}.

$E_1^{-1}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, $E_2^{-1}=\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}$, $E_3^{-1}=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$

d) Using the above results, express A as an explicit product of elementary matrices.

$$A=E_1^{-1}E_2^{-1}E_3^{-1}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}.$$