136.130 Midterm
Brief Solutions

1(a) $-2((2,1,0)\cdot(3,1,4))+\|3(2,1,0)+2(1,0,-2)-(3,1,4)\|^2=-14+93=79$

1(b) $\mathbf{u}\times\mathbf{v}=(-2,4,-1)$

1(c) $\dfrac{\mathbf{u}\times\mathbf{v}}{\|\mathbf{u}\times\mathbf{v}\|}=\dfrac{(-2,4,-1)}{\sqrt{21}}$

1(d) $\cos(\text{angle between } \mathbf{u} \text{ and } \mathbf{w})=\dfrac{\mathbf{u}\cdot\mathbf{w}}{\|\mathbf{u}\|\,\|\mathbf{w}\|}=\dfrac{7}{\sqrt{130}}$.

2.(a) Solve the system of all four given equations to get (2,1,3) as the point of intersection.

2(b) The vector $\mathbf{v}=(1,4,-1)$ is parallel to the line we want. So its parametric equations are

$$x=5+t$$
$$y=0+4t$$
$$z=4-t$$

2(c) The vector $\mathbf{n}_\Pi=(3,2,-1)$ is perpendicular to Π and so it must be parallel to the plane we want. Any vector $\mathbf{n}$ perpendicular to $\mathbf{n}_\Pi$ will give rise to a plane perpendicular to Π. We can choose $\mathbf{n}$ by making the dot product $\mathbf{n}\,.\,\mathbf{n}_\Pi=0$. One solution is $\mathbf{n}=(0,1,2)$. The plane perpendicular to this $\mathbf{n}=(0,1,2)$ and passing through P is $(\mathbf{x}-(5,0,4))\cdot(0,1,2)=0$, which in standard form becomes $y+2z=8$. (Note again: this is not the only correct solution.)

3(a) $\left[\begin{array}{ccc|c} 2 & 3 & 5 & 4 \\ 4 & 7 & 13 & 6 \end{array}\right]$

3(b) The RREF of the above matrix is $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 3 & -2 \end{array}\right]$. After solving the associated system we get $x=5+2t,\ y=-2-3t,\ z=t$ where t ranges through the set of all real numbers.

4(a) The RREF of the given augmented matrix is $\left[\begin{array}{ccccc|c} 1 & 5 & 0 & 4 & 0 & -14 \\ 0 & 0 & 1 & 2 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right]$

4(b) The solution of the associated system is

$$\begin{aligned} x_1 &= -14 - 5t - 4s \\ x_2 &= t \\ x_3 &= -5 - 2s \\ x_4 &= s \\ x_5 &= 3 \end{aligned}$$

where t and s range through the set of all real numbers.

4(c) Set $x_2 = 2$ in the above solution to get

$$\begin{aligned} x_1 &= -24 - 4s \\ x_2 &= 2 \\ x_3 &= -5 - 2s \\ x_4 &= s \\ x_5 &= 3 \end{aligned}$$

where s ranges through the set of all real numbers.

5(a) $AB + 2C = \begin{bmatrix} -3 & 0 \\ 15 & 25 \end{bmatrix}$

5(b) $BC + CA$ is not defined because BC is of size 3x2, while CA is not.

5(c) $C(A - B^T) = \begin{bmatrix} 4 & 1 & -1 \\ 12 & -11 & 19 \end{bmatrix}$.

5(d) $(A^T - B)C^T = [(A - B^T)]^T C^T = [(A - B^T)C]^T = \begin{bmatrix} 4 & 1 & -1 \\ 12 & -11 & 19 \end{bmatrix}^T = \begin{bmatrix} 4 & 12 \\ 1 & -11 \\ -1 & 19 \end{bmatrix}$,

where in the second to the last step we have used 5(c). Alternatively, do all the operations directly.

6(a) Apply row reduction to the matrix $\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ -1 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$ to get the following RREF matrix: $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -8 & 2 & -9 \\ 0 & 1 & 0 & -4 & 1 & -4 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$. It follows that $A^{-1} = \begin{bmatrix} -8 & 2 & -9 \\ -4 & 1 & -4 \\ 1 & 0 & 1 \end{bmatrix}$.

6(b) $A\mathbf{x} = \mathbf{b}$ is equivalent to $\mathbf{x} = A^{-1}\mathbf{b}$ and we perform the multiplication on the left-hand side to get $\mathbf{x} = \begin{bmatrix} -31 \\ -14 \\ 4 \end{bmatrix}$, or $x = -31,\ y = -14,\ z = 4$.