THE UNIVERSITY OF MANITOBA

DATE: October 28, 2002 MIDTERM EXAMINATION

DEPARTMENT & COURSE NO: 136.130 TIME: 1 hour

EXAMINATION: Vector Geometry & Linear Algebra EXAMINERS: Various

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Values

[10] 1. Consider the following system of equations:

$$\begin{aligned} 2x &+ 4y + z = 11 \\ x &+ 2y + 2z = 13 \\ 3x &+ 6y + 3z = 24 \end{aligned}$$

(a) Write down the augmented matrix for this system.

$$\left[\begin{array}{ccc|c} 2 & 4 & 1 & 11 \\ 1 & 2 & 2 & 13 \\ 3 & 6 & 3 & 24 \end{array}\right]$$

(b) Find the reduced row-echelon form of this matrix.

Row reduce $\left[\begin{array}{ccc|c} 2 & 4 & 1 & 11 \\ 1 & 2 & 2 & 13 \\ 3 & 6 & 3 & 24 \end{array}\right]$ to get $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

(c) Solve the system; present your answer in parametric form.

The system associated to the last augmented matrix above is

$$\begin{aligned} x + 2y &= 3 \\ z &= 5 \\ 0 &= 0 \end{aligned}$$

Choose $y=t$ to get the solution in the parametric form: x=-2t+3, y=t, z=5.

Values

[10] 2. (a) Let $A = \begin{matrix} 1 & 4 & 3 \\ 2 & 1 & 5 \end{matrix}$, $B = \begin{matrix} 2 & 3 & 6 \\ 1 & 1 & 3 \\ 1 & 2 & 2 \end{matrix}$, $C = \begin{matrix} 10 & 4 \\ 1 & 1 \\ 2 & 0 \end{matrix}$

Find, if possible, $A(B \quad B^T)$, $B(A^T + C)$, $(A^T + C)B$, $B(A + C^T)$ and $(A + C^T)B^T$.

1. $A(B \quad B^T) = \begin{matrix} 7 & 1 & 1 \\ 23 & 9 & 11 \end{matrix}$.

2. $B(A^T + C) = \begin{matrix} 7 & 26 \\ 31 & 17 \\ 9 & 8 \end{matrix}$.

3. $(A^T + C)B$ is not doable.

4. $B(A + C^T)$ is not doable.

5. Note that $(A + C^T)B^T$ is the transpose of $B(A^T + C)$. So,

$$(A + C^T)B^T = [B(A^T + C)]^T = \begin{matrix} 7 & 26 \\ 31 & 17 \\ 9 & 8 \end{matrix}^T = \begin{matrix} 7 & 31 & 9 \\ 26 & 17 & 8 \end{matrix}.$$

(b) Let A be a 2 3 matrix, C a 4 5 matrix, and suppose that $M = ABC$.

What are the sizes of the matrices M and B?

Since A is 2x3 and C is 4x5, it follows that B is 3x4 and that M is 2x5.

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[10] 3. Let $A = \begin{matrix} 1 & 0 & 1 \\ 2 & 2 & 3 \\ 2 & 1 & 2 \end{matrix}$.

(a) Find A^{1}.

Use row reduction or AdjA to get that $A^{1} = \begin{matrix} 1 & 1 & 2 \\ 2 & 0 & 1 \\ 2 & 1 & 2 \end{matrix}$.

(b) Use your answer in (a) to solve the system $A \begin{matrix} p \\ q \\ r \end{matrix} = \begin{matrix} 3 \\ 2 \\ 4 \end{matrix}$.

(Full marks will not be awarded for solutions that do not use (a)).

$$\begin{matrix} p \\ q \\ r \end{matrix} = \begin{matrix} 1 & 1 & 2 \\ 2 & 0 & 1 \\ 2 & 1 & 2 \end{matrix} \begin{matrix} 3 \\ 2 \\ 4 \end{matrix} = \begin{matrix} 9 \\ 10 \\ 12 \end{matrix}$$

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[10] 4. (a) Which of the following matrices are elementary matrices:

$$A = \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}, \quad B = \begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}. \quad C = \begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{matrix},$$

$$D = \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix}, \quad E = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}, \quad F = \begin{matrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{matrix}.$$

B, C and E are elementary; the other matrices are not elementary.

(b) Let $C = \begin{matrix} 1 & 2 \\ 4 & 7 \end{matrix}$. Write C as a product of elementary matrices.

Row reduce: $\begin{matrix} 1 & 2 \\ 4 & 7 \end{matrix} \xrightarrow{(4)(1st)\ to\ 2nd} \begin{matrix} 1 & 2 \\ 0 & 1 \end{matrix} \xrightarrow{(\ 1)1st} \begin{matrix} 1 & 2 \\ 0 & 1 \end{matrix} \xrightarrow{(2)2nd\ to\ 1st} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$. The first matrix is C, the last is the identity. Call the second matrix A_1. The third matrix is already elementary; call it E_1. Note that we can get A_1 from E_1 by multiplying the 1st row by –1. So, $A_1 = E_2 \begin{matrix} 1 & 2 \\ 0 & 1 \end{matrix}$, where we get E_2 by multiplying the first row of I by –1; so $E_2 = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$. Further, we can get C from A_1 by multiplying the 1st row by (-4) and adding that to the second row. So, $C = E_3 A_1$, where we get E_3 by multiplying the 1st row of I by (-4) and adding that to the second row; so $E_3 = \begin{matrix} 1 & 0 \\ 4 & 1 \end{matrix}$. Summarizing: $C = E_3 A_1 = E_3 E_2 E_1 = \begin{matrix} 1 & 0 \\ 4 & 1 \end{matrix} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \begin{matrix} 1 & 2 \\ 0 & 1 \end{matrix}$

Note: this is NOT the only correct answer.

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Values

[10] 5. (a) Compute $\begin{vmatrix} 1 & 2 & 1 \\ 3 & 6 & 5 \\ 2 & 1 & 3 \end{vmatrix}$ by using row operations to put the matrix in upper-triangular form. (Full marks will not be awarded for any other method.)

Start with $\begin{matrix} 1 & 2 & 1 \\ 3 & 6 & 5 \\ 2 & 1 & 3 \end{matrix}$ and row reduce a bit: $\begin{matrix} 1 & 2 & 1 \\ 3 & 6 & 5 \\ 2 & 1 & 3 \end{matrix} \xrightarrow{(\ 3)\text{1st to 2nd and } (-2)\text{1st to 3rd}} \begin{matrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 5 & 1 \end{matrix}$. Note that the two row operations we have applied so far do not affect the determinant. One more step: $\begin{matrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 5 & 1 \end{matrix} \xrightarrow{\text{switch 2nd and 3rd}} \begin{matrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 2 \end{matrix}$. This row operation affect the determinant by a factor of –1. So the determinant we want is $(\ 1)\det \begin{matrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 2 \end{matrix}$. But since this matrix is triangular, we have

$$(\ 1)\det \begin{matrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 2 \end{matrix} = (\ 1)(1)(\ 5)(2) = \ 10.$$

(b) Given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$, find $\begin{vmatrix} 2a & b & c \\ 2(a+d) & b+e & c+f \\ 2(3g+a) & 3h+b & 3i+c \end{vmatrix}$.

$$\begin{vmatrix} 2a & b & c \\ 2(a+d) & b+e & c+f \\ 2(3g+a) & 3h+b & 3i+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ (a+d) & b+e & c+f \\ (3g+a) & 3h+b & 3i+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ d & e & f \\ (3g+a) & 3h+b & 3i+c \end{vmatrix} =$$

$$= 2\begin{vmatrix} a & b & c \\ d & e & f \\ 3g & 3h & 3i \end{vmatrix} = (2)(3)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (6)(7) = 42$$

(c) A, B, C are $4 \quad 4$ matrices with determinants $\det A = \ 2$, $\det B = 3$ and $\det C = 5$. Find $\det\left(\ 3A^2B^tC^{\ 1}\right)$.

$$\det\left(\ 3A^2B^tC^{\ 1}\right) = (\ 3)^4[\det(A)]^2(\det B^T)(\det C^{\ 1}) = (\ 3)^4(\ 2)^2(3)(1/5).$$

Values

[10] 6. (a) Use Cramer's rule to find the value of x_3 in the system:

$$\begin{array}{rcrcrcr} 3x_1 & & & + & x_3 & = & 2 \\ 4x_1 & + & 2x_2 & + & 4x_3 & = & 4 \\ & & x_2 & & x_3 & = & 0 \end{array}$$

The coefficient matrix is $A = \begin{matrix} 3 & 0 & 1 \\ 4 & 2 & 4 \\ 0 & 1 & 1 \end{matrix}$ and $\det A = \ 14$. The matrix A_3 we need to solve for x_3 is $A_3 = \begin{matrix} 3 & 0 & 2 \\ 4 & 2 & 4 \\ 0 & 1 & 0 \end{matrix}$, and it has the determinant of –4. So, $x_3 = \frac{\det A_3}{\det A} = \frac{4}{14} = \frac{2}{7}$.

(b) Suppose $A = \begin{matrix} 1 & 0 & a \\ 2 & 2 & 3 \\ 1 & 1 & 2 \end{matrix}$ and $\text{Adj } A = \begin{matrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & b & c \end{matrix}$.

Find the values of a, b, and c, and use Adj A to find $A^{\ 1}$

We read from AdjA that the cofactor $C_{2,2}$ is 1. On the other hand, we see from the matrix A that $C_{2,2} = (\ \ 1)^{2+2}\begin{vmatrix} 1 & a \\ 1 & 2 \end{vmatrix}$. So, $1 = (\ \ 1)^{2+2}\begin{vmatrix} 1 & a \\ 1 & 2 \end{vmatrix}$, from where we find $a = 1$.

Notice that $b = C_{2,3}$. We then compute $C_{2,3}$ from the matrix A to get that b=1.
Finally, notice that $c = C_{3,3}$ and we find from A that c=-2.

Now that we know a we compute $\det A = \ 1$. Since $A^{\ 1} = \frac{1}{\det A} AdjA$, we have

$$A^{\ 1} = \frac{1}{\ 1}\begin{matrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{matrix}.$$