Mathematics MATH1300
Vector Geometry and Linear Algebra
Midterm Examination
October 26 2015, 5:00–6:00pm

Name (Print in ink): ______________________________

Student number (in ink): ___________________________

Signature (in ink): ________________________________
(I understand that cheating is a serious offense.)

Indicate your instructor by checking the appropriate box below:

☐	S. Kalajdzievski	MWF 9:30	208 Armes
☐	S. Zadeh	TTh 8:30	204 Armes
☐	M. Virgilio	MWF 11:30	EITC E3 270
☐	M. Doob	TTh 11:30	208 Armes
☐	Other		

Instructions:

This is a one hour examination. **Show your work clearly.**

No texts, notes or other aids are permitted. Calculators, cell phones and electronic translators in particular are disallowed.

Answer all questions on the exam paper in the space provided beneath the question. If more room is needed, you may continue on the reverse side, but **clearly indicate that your work continues there**.

There are seven problems on this exam. The value of each is indicated at the beginning of the problem.

Question	Points	Score
1	20	
2	7	
3	20	
4	6	
5	7	
6	20	
7	20	
Total	100	

1. (20%) Consider the following system of linear equations:

$$\begin{aligned} x_1 + 3x_2 + + 6x_4 &= 4 \\ x_1 + 3x_2 + 4x_3 - 2x_4 &= 8 \end{aligned}$$

(a) Give the augmented matrix of the system.

(b) Put the augmented matrix in reduced row echelon form.

(c) Give **all** solutions to the system of equations.

(d) Find a solution to the above system with $x_2 = -4$ and $x_4 = -1$.

Solution. (a) The augmented matrix is $\left[\begin{array}{cccc|c} 1 & 3 & 0 & 6 & 4 \\ 1 & 3 & 4 & -2 & 8 \end{array}\right]$.

(b) $\left[\begin{array}{cccc|c} 1 & 3 & 0 & 6 & 4 \\ 0 & 0 & 4 & -8 & 4 \end{array}\right]$ $R_2 \leftarrow R_2 - 3R_1$

$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 6 & 4 \\ 0 & 0 & 1 & -2 & 1 \end{array}\right]$ $R_2 \leftarrow \frac{1}{4}R_2$

(c) This reduced row echelon form gives the general solution is

$$(x_1, x_2, x_3, x_4) = (4 - 3u - 6v, u, 1 + 2v, v)$$

(d) When $x_2 = u = -4$ and $x_4 = -1 = v$, the solution is

$$(x_1, x_2, x_3, x_4) = (22, -4, -1, -1)$$

2. (7%) In the following system a and b are constants

$$\begin{array}{rcrcl} x & + & y & = & 1-b \\ 3x & + & 4ay & = & 3b \end{array}$$

(a) Find all values of a and b such that the system above has no solutions.

(b) Find all values of a and b such that the system above has infinitely many solutions.

(c) Find all values of a and b such that the system above has exactly one solution.

Solution. The augmented matrix is $\left[\begin{array}{cc|c} 1 & 1 & 1-b \\ 3 & 4a & 3b \end{array}\right]$, which in one step reduces to $\left[\begin{array}{cc|c} 1 & 1 & 1-b \\ 0 & 4a-3 & -3+6b \end{array}\right]$. If $4a-3 \neq 0$, then divide the second row by it and that row will then contain a leading one. This makes the solution is unique. If $4a-3=0$, then $-3+6b=0$ makes the last row all zeros (infinite number of solutions). If $-3+6b \neq 0$, then there is no solution.

in summation:

$$\begin{array}{rl} \text{no solution:} & a = \frac{3}{4} \text{ and } b \neq \frac{1}{2} \\ \text{one solution:} & a \neq \frac{3}{4} \\ \text{infinite solutions:} & a = \frac{3}{4} \text{ and } b = \frac{1}{2} \end{array}$$

3. (20%) Let $A = \begin{pmatrix} 4 & -1 \\ 3 & -1 \end{pmatrix}$. Express A^{-1} as a product of elementary matrices. Show all your work.

Solution

Matrix	elementary row op	matrix
$\begin{bmatrix} 1 & -\frac{1}{4} \\ 3 & -1 \end{bmatrix}$	$R_1 \leftarrow \frac{1}{4}R_1$	$\begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & -\frac{1}{4} \\ 0 & -\frac{1}{4} \end{bmatrix}$	$R_2 \leftarrow R_2 - 3R_1$	$\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & -\frac{1}{4} \\ 0 & 1 \end{bmatrix}$	$R_2 \leftarrow -4R_2$	$\begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$R_1 \leftarrow R_1 + \frac{1}{4}R_2$	$\begin{bmatrix} 1 & \frac{1}{4} \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & \frac{1}{4} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix} = A^{-1}$$

Note: there is a shorter solution.

4. (6%) Using Cramer's rule, find all solutions to the following system of linear equations (**Cramer's rule must be used**):

$$\begin{aligned} 3x + 4y &= -2 \\ 5x + 3y &= 4 \end{aligned}$$

Solution.

$$\det \begin{bmatrix} 3 & 4 \\ 5 & 3 \end{bmatrix} = -11, \det \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix} = 22, \text{ and } \det \begin{bmatrix} -2 & 4 \\ 4 & 3 \end{bmatrix} = -22,$$

and so $x = 2$ and $y = -2$.

5. (7%) Let

$$A = \begin{bmatrix} 0 & 3 & 1 & 2 & 1 \\ 0 & -2 & 0 & 0 & 0 \\ -2 & 2 & -1 & 2 & -1 \\ 0 & -2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 & 3 \end{bmatrix}$$

Evaluate the determinant of A.

Solution.

$$\begin{aligned} \det A &= \det \begin{bmatrix} 0 & 3 & 1 & 2 & 1 \\ 0 & -2 & 0 & 0 & 0 \\ -2 & 2 & -1 & 2 & -1 \\ 0 & -2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 & 3 \end{bmatrix} \\ &= -2 \det \begin{bmatrix} 3 & 1 & 2 & 1 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 1 & 1 \\ 1 & 0 & 2 & 3 \end{bmatrix} = -4 \det \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix} \\ &= -4 \det \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = -4 \end{aligned}$$

6. (20%) Let $A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(a) Compute A^{-1} by putting A in reduced row echelon form. Indicate the elementary row operation that you are using at each step.

(b) Let C be the cofactor matrix of A. Here is part of it:

$$C = \begin{bmatrix} 2 & 0 & 0 \\ x & 2 & 0 \\ -3 & 0 & y \end{bmatrix}$$

Compute x and y, the remaining entries (show your work).

(c) Compute the adjoint of A.

(d) Compute A^{-1} using the adjoint matrix $\text{adj}(A)$.

Solution.

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] R_1 \leftarrow R_1 - 4R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & -4 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] R_3 \leftarrow \tfrac{1}{2} R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & -4 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right] R_1 \leftarrow R_1 - 3R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -4 & -\frac{3}{2} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right]$$

$x = -\det \begin{bmatrix} 4 & 3 \\ 0 & 2 \end{bmatrix} = -8$ and $y = \det \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = 1$, and so

$C = \begin{bmatrix} 2 & 0 & 0 \\ -8 & 2 & 0 \\ -3 & 0 & 1 \end{bmatrix}$ and $\text{adj}\, A = C^T$

$\det A = 2$ which makes $A^{-1} = \frac{1}{2} \text{adj}\, A$ from which follows $A^{-1} = \begin{bmatrix} 1 & -4 & -\frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$.

7. (20%) Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$. In each case, evaluate the expression if possible. If not possible, explain why,

(a) AB

(b) BA

(c) $A^T B^T$

(d) $B^T A^T$

(e) $B + B^T$

Solution.

(a) AB is not defined since A has three columns and B has two rows

(b) $BA = \begin{bmatrix} -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix}$

(c) $A^T B^T = (BA)^T = \begin{bmatrix} -1 & 2 \\ 1 & -2 \\ -2 & 4 \end{bmatrix}$

(d) $B^T A^T$ is not well defined.

(e) $B + B^T = \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix}$

This page is for scratch work. It will not be looked at for marking purposes.