7.1

EIGENVALUES AND
EIGENVECTORS

In Section 2.3 we introduced the concepts of eigenvalue and eigenvector. In this section we will study those ideas in more detail to set the stage for applications of them in later sections.

We begin with a review of some concepts that were mentioned in Sections 2.3 and 4.3.

## DEFINITION

If $A$ is an $n \times n$ matrix, then a nonzero vector $\mathbf{x}$ in $R^{n}$ is called an eigenvector of $A$ if $A \mathbf{x}$ is a scalar multiple of $\mathbf{x}$; that is, if

$$
A \mathbf{x}=\lambda \mathbf{x}
$$

for some scalar $\lambda$. The scalar $\lambda$ is called an eigenvalue of $A$, and $\mathbf{x}$ is said to be an eigenvector of $A$ corresponding to $\lambda$.

In $R^{2}$ and $R^{3}$, multiplication by $A$ maps each eigenvector $\mathbf{x}$ of $A$ (if any) onto the same line through the origin as $\mathbf{x}$. Depending on the sign and the magnitude of the eigenvalue $\lambda$ corresponding to $\mathbf{x}$, the linear operator $A \mathbf{x}=\lambda \mathbf{x}$ compresses or stretches $\mathbf{x}$ by a factor of $\lambda$, with a reversal of direction in the case where $\lambda$ is negative (Figure 7.1.1).

(a) $0 \leq \lambda \leq 1$

(b) $\lambda \geq 1$

(c) $-1 \leq \lambda \leq 0$

(d) $\lambda \leq-1$

Figure 7.1.1

## EXAMPLE 1 Eigenvector of a $2 \times 2$ Matrix

The vector $\mathbf{x}=\left[\begin{array}{l}1 \\ 2\end{array}\right]$ is an eigenvector of

$$
A=\left[\begin{array}{rr}
3 & 0 \\
8 & -1
\end{array}\right]
$$

corresponding to the eigenvalue $\lambda=3$, since

$$
A \mathbf{x}=\left[\begin{array}{rr}
3 & 0 \\
8 & -1
\end{array}\right]\left[\begin{array}{l}
1 \\
2
\end{array}\right]=\left[\begin{array}{l}
3 \\
6
\end{array}\right]=3 \mathbf{x}
$$

To find the eigenvalues of an $n \times n$ matrix $A$, we rewrite $A \mathbf{x}=\lambda \mathbf{x}$ as

$$
A \mathbf{x}=\lambda I \mathbf{x}
$$

or, equivalently,

$$
\begin{equation*}
(\lambda I-A) \mathbf{x}=\mathbf{0} \tag{1}
\end{equation*}
$$

For $\lambda$ to be an eigenvalue, there must be a nonzero solution of this equation. By Theorem 6.4.5, Equation (1) has a nonzero solution if and only if

$$
\operatorname{det}(\lambda I-A)=0
$$

This is called the characteristic equation of $A$; the scalars satisfying this equation are the eigenvalues of $A$. When expanded, the determinant $\operatorname{det}(\lambda I-A)$ is always a polynomial $p$ in $\lambda$, called the characteristic polynomial of $A$.

It can be shown (Exercise 15) that if $A$ is an $n \times n$ matrix, then the characteristic polynomial of $A$ has degree $n$ and the coefficient of $\lambda^{n}$ is 1 ; that is, the characteristic polynomial $p(\lambda)$ of an $n \times n$ matrix has the form

$$
p(\lambda)=\operatorname{det}(\lambda I-A)=\lambda^{n}+c_{1} \lambda^{n-1}+\cdots+c_{n}
$$

It follows from the Fundamental Theorem of Algebra that the characteristic equation

$$
\lambda^{n}+c_{1} \lambda^{n-1}+\cdots+c_{n}=0
$$

has at most $n$ distinct solutions, so an $n \times n$ matrix has at most $n$ distinct eigenvalues.
The reader may wish to review Example 6 of Section 2.3, where we found the eigenvalues of a $2 \times 2$ matrix by solving the characteristic equation. The following example involves a $3 \times 3$ matrix.

## EXAMPLE 2 Eigenvalues of a $3 \times 3$ Matrix

Find the eigenvalues of

$$
A=\left[\begin{array}{rrr}
0 & 1 & 0 \\
0 & 0 & 1 \\
4 & -17 & 8
\end{array}\right]
$$

## Solution

The characteristic polynomial of $A$ is

$$
\operatorname{det}(\lambda I-A)=\operatorname{det}\left[\begin{array}{rrc}
\lambda & -1 & 0 \\
0 & \lambda & -1 \\
-4 & 17 & \lambda-8
\end{array}\right]=\lambda^{3}-8 \lambda^{2}+17 \lambda-4
$$

The eigenvalues of $A$ must therefore satisfy the cubic equation

$$
\begin{equation*}
\lambda^{3}-8 \lambda^{2}+17 \lambda-4=0 \tag{2}
\end{equation*}
$$

To solve this equation, we shall begin by searching for integer solutions. This task can be greatly simplified by exploiting the fact that all integer solutions (if there are any) to a polynomial equation with integer coefficients

$$
\lambda^{n}+c_{1} \lambda^{n-1}+\cdots+c_{n}=0
$$

must be divisors of the constant term, $c_{n}$. Thus, the only possible integer solutions of (2) are the divisors of -4 , that is, $\pm 1, \pm 2, \pm 4$. Successively substituting these values in (2) shows that $\lambda=4$ is an integer solution. As a consequence, $\lambda-4$ must be a factor of the
left side of (2). Dividing $\lambda-4$ into $\lambda^{3}-8 \lambda^{2}+17 \lambda-4$ shows that (2) can be rewritten as

$$
(\lambda-4)\left(\lambda^{2}-4 \lambda+1\right)=0
$$

Thus the remaining solutions of (2) satisfy the quadratic equation

$$
\lambda^{2}-4 \lambda+1=0
$$

which can be solved by the quadratic formula. Thus the eigenvalues of $A$ are

$$
\lambda=4, \quad \lambda=2+\sqrt{3}, \quad \text { and } \quad \lambda=2-\sqrt{3}
$$

REMARK In practical problems, the matrix $A$ is usually so large that computing the characteristic equation is not practical. As a result, other methods are used to obtain eigenvalues.

## EXAMPLE 3 Eigenvalues of an Upper Triangular Matrix

Find the eigenvalues of the upper triangular matrix

$$
A=\left[\begin{array}{cccc}
a_{11} & a_{12} & a_{13} & a_{14} \\
0 & a_{22} & a_{23} & a_{24} \\
0 & 0 & a_{33} & a_{34} \\
0 & 0 & 0 & a_{44}
\end{array}\right]
$$

## Solution

Recalling that the determinant of a triangular matrix is the product of the entries on the main diagonal (Theorem 2.1.3), we obtain

$$
\begin{aligned}
\operatorname{det}(\lambda I-A) & =\operatorname{det}\left[\begin{array}{cccc}
\lambda-a_{11} & -a_{12} & -a_{13} & -a_{14} \\
0 & \lambda-a_{22} & -a_{23} & -a_{24} \\
0 & 0 & \lambda-a_{33} & -a_{34} \\
0 & 0 & 0 & \lambda-a_{44}
\end{array}\right] \\
& =\left(\lambda-a_{11}\right)\left(\lambda-a_{22}\right)\left(\lambda-a_{33}\right)\left(\lambda-a_{44}\right)
\end{aligned}
$$

Thus, the characteristic equation is

$$
\left(\lambda-a_{11}\right)\left(\lambda-a_{22}\right)\left(\lambda-a_{33}\right)\left(\lambda-a_{44}\right)=0
$$

and the eigenvalues are

$$
\lambda=a_{11}, \quad \lambda=a_{22}, \quad \lambda=a_{33}, \quad \lambda=a_{44}
$$

which are precisely the diagonal entries of $A$.

The following general theorem should be evident from the computations in the preceding example.

## Complex <br> Eigenvalues

## EXAMPLE 4 Eigenvalues of a Lower Triangular Matrix

By inspection, the eigenvalues of the lower triangular matrix

$$
A=\left[\begin{array}{rrr}
\frac{1}{2} & 0 & 0 \\
-1 & \frac{2}{3} & 0 \\
5 & -8 & -\frac{1}{4}
\end{array}\right]
$$

are $\lambda=\frac{1}{2}, \lambda=\frac{2}{3}$, and $\lambda=-\frac{1}{4}$.

It is possible for the characteristic equation of a matrix with real entries to have complex solutions. In fact, because the eigenvalues of an $n \times n$ matrix are the roots of a polynomial of precise degree $n$, every $n \times n$ matrix has exactly $n$ eigenvalues if we count them as we count the roots of a polynomial (meaning that they may be repeated, and may occur in complex conjugate pairs). For example, the characteristic polynomial of the matrix

$$
A=\left[\begin{array}{rr}
-2 & -1 \\
5 & 2
\end{array}\right]
$$

is

$$
\operatorname{det}(\lambda I-A)=\operatorname{det}\left[\begin{array}{cc}
\lambda+2 & 1 \\
-5 & \lambda-2
\end{array}\right]=\lambda^{2}+1
$$

so the characteristic equation is $\lambda^{2}+1=0$, the solutions of which are the imaginary numbers $\lambda=i$ and $\lambda=-i$. Thus we are forced to consider complex eigenvalues, even for real matrices. This, in turn, leads us to consider the possibility of complex vector spaces-that is, vector spaces in which scalars are allowed to have complex values. Such vector spaces will be considered in Chapter 10. For now, we will allow complex eigenvalues, but we will limit our discussion of eigenvectors to the case of real eigenvalues.

The following theorem summarizes our discussion thus far.

THEOREM 7.1.2

Finding Eigenvectors and Bases for Eigenspaces

## Equivalent Statements

If $A$ is an $n \times n$ matrix and $\lambda$ is a real number, then the following are equivalent.
(a) $\lambda$ is an eigenvalue of $A$.
(b) The system of equations $(\lambda I-A) \mathbf{x}=\mathbf{0}$ has nontrivial solutions.
(c) There is a nonzero vector $\mathbf{x}$ in $R^{n}$ such that $A \mathbf{x}=\lambda \mathbf{x}$.
(d) $\lambda$ is a solution of the characteristic equation $\operatorname{det}(\lambda I-A)=0$.

Now that we know how to find eigenvalues, we turn to the problem of finding eigenvectors. The eigenvectors of $A$ corresponding to an eigenvalue $\lambda$ are the nonzero vectors $\mathbf{x}$ that satisfy $A \mathbf{x}=\lambda \mathbf{x}$. Equivalently, the eigenvectors corresponding to $\lambda$ are the nonzero vectors in the solution space of $(\lambda I-A) \mathbf{x}=\mathbf{0}$-that is, in the null space of $\lambda I-A$. We call this solution space the eigenspace of $A$ corresponding to $\lambda$.

## EXAMPLE 5 Eigenvectors and Bases for Eigenspaces

Find bases for the eigenspaces of

$$
A=\left[\begin{array}{rrr}
0 & 0 & -2 \\
1 & 2 & 1 \\
1 & 0 & 3
\end{array}\right]
$$

## Solution

The characteristic equation of matrix $A$ is $\lambda^{3}-5 \lambda^{2}+8 \lambda-4=0$, or, in factored form, $(\lambda-1)(\lambda-2)^{2}=0$ (verify); thus the eigenvalues of $A$ are $\lambda_{1}=1$ and $\lambda_{2,3}=2$, so there are two eigenspaces of $A$.

By definition,

$$
\mathbf{x}=\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right]
$$

is an eigenvector of $A$ corresponding to $\lambda$ if and only if $\mathbf{x}$ is a nontrivial solution of $(\lambda I-A) \mathbf{x}=\mathbf{0}$-that is, of

$$
\left[\begin{array}{ccc}
\lambda & 0 & 2  \tag{3}\\
-1 & \lambda-2 & -1 \\
-1 & 0 & \lambda-3
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right]
$$

If $\lambda=2$, then (3) becomes

$$
\left[\begin{array}{rrr}
2 & 0 & 2 \\
-1 & 0 & -1 \\
-1 & 0 & -1
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right]
$$

Solving this system using Gaussian elimination yields (verify)

$$
x_{1}=-s, \quad x_{2}=t, \quad x_{3}=s
$$

Thus, the eigenvectors of $A$ corresponding to $\lambda=2$ are the nonzero vectors of the form

$$
\mathbf{x}=\left[\begin{array}{r}
-s \\
t \\
s
\end{array}\right]=\left[\begin{array}{r}
-s \\
0 \\
s
\end{array}\right]+\left[\begin{array}{l}
0 \\
t \\
0
\end{array}\right]=s\left[\begin{array}{r}
-1 \\
0 \\
1
\end{array}\right]+t\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right]
$$

Since

$$
\left[\begin{array}{r}
-1 \\
0 \\
1
\end{array}\right] \quad \text { and } \quad\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right]
$$

are linearly independent, these vectors form a basis for the eigenspace corresponding to $\lambda=2$.

If $\lambda=1$, then (3) becomes

$$
\left[\begin{array}{rrr}
1 & 0 & 2 \\
-1 & -1 & -1 \\
-1 & 0 & -2
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right]
$$

Solving this system yields (verify)

$$
x_{1}=-2 s, \quad x_{2}=s, \quad x_{3}=s
$$

Thus the eigenvectors corresponding to $\lambda=1$ are the nonzero vectors of the form

$$
\left[\begin{array}{r}
-2 s \\
s \\
s
\end{array}\right]=s\left[\begin{array}{r}
-2 \\
1 \\
1
\end{array}\right] \text { so that }\left[\begin{array}{r}
-2 \\
1 \\
1
\end{array}\right]
$$

is a basis for the eigenspace corresponding to $\lambda=1$.

THEOREM 7.1.3

Eigenvalues and Invertibility

THEOREM 7.1.4

Once the eigenvalues and eigenvectors of a matrix $A$ are found, it is a simple matter to find the eigenvalues and eigenvectors of any positive integer power of $A$; for example, if $\lambda$ is an eigenvalue of $A$ and $\mathbf{x}$ is a corresponding eigenvector, then

$$
A^{2} \mathbf{x}=A(A \mathbf{x})=A(\lambda \mathbf{x})=\lambda(A \mathbf{x})=\lambda(\lambda \mathbf{x})=\lambda^{2} \mathbf{x}
$$

which shows that $\lambda^{2}$ is an eigenvalue of $A^{2}$ and that $\mathbf{x}$ is a corresponding eigenvector. In general, we have the following result.

If $k$ is a positive integer, $\lambda$ is an eigenvalue of a matrix $A$, and $\mathbf{x}$ is a corresponding eigenvector, then $\lambda^{k}$ is an eigenvalue of $A^{k}$ and $\mathbf{x}$ is a corresponding eigenvector.

## EXAMPLE 6 Using Theorem 7.1.3

In Example 5 we showed that the eigenvalues of

$$
A=\left[\begin{array}{rrr}
0 & 0 & -2 \\
1 & 2 & 1 \\
1 & 0 & 3
\end{array}\right]
$$

are $\lambda=2$ and $\lambda=1$, so from Theorem 7.1.3, both $\lambda=2^{7}=128$ and $\lambda=1^{7}=1$ are eigenvalues of $A^{7}$. We also showed that

$$
\left[\begin{array}{r}
-1 \\
0 \\
1
\end{array}\right] \text { and }\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right]
$$

are eigenvectors of $A$ corresponding to the eigenvalue $\lambda=2$, so from Theorem 7.1.3, they are also eigenvectors of $A^{7}$ corresponding to $\lambda=2^{7}=128$. Similarly, the eigenvector

$$
\left[\begin{array}{r}
-2 \\
1 \\
1
\end{array}\right]
$$

of $A$ corresponding to the eigenvalue $\lambda=1$ is also an eigenvector of $A^{7}$ corresponding to $\lambda=1^{7}=1$.

The next theorem establishes a relationship between the eigenvalues and the invertibility of a matrix.

A square matrix $A$ is invertible if and only if $\lambda=0$ is not an eigenvalue of $A$.

Proof Assume that $A$ is an $n \times n$ matrix and observe first that $\lambda=0$ is a solution of the characteristic equation

$$
\lambda^{n}+c_{1} \lambda^{n-1}+\cdots+c_{n}=0
$$

if and only if the constant term $c_{n}$ is zero. Thus it suffices to prove that $A$ is invertible if and only if $c_{n} \neq 0$. But

$$
\operatorname{det}(\lambda I-A)=\lambda^{n}+c_{1} \lambda^{n-1}+\cdots+c_{n}
$$

or, on setting $\lambda=0$,

$$
\operatorname{det}(-A)=c_{n} \quad \text { or } \quad(-1)^{n} \operatorname{det}(A)=c_{n}
$$

It follows from the last equation that $\operatorname{det}(A)=0$ if and only if $c_{n}=0$, and this in turn implies that $A$ is invertible if and only if $c_{n} \neq 0$.

## EXAMPLE 7 Using Theorem 7.1.4

The matrix $A$ in Example 5 is invertible since it has eigenvalues $\lambda=1$ and $\lambda=2$, neither of which is zero. We leave it for the reader to check this conclusion by showing that $\operatorname{det}(A) \neq 0$.

## Summary

THEOREM 7.1.5

Theorem 7.1.4 enables us to add an additional result to Theorem 6.4.5.

## Equivalent Statements

If $A$ is an $n \times n$ matrix, and if $T_{A}: R^{n} \rightarrow R^{n}$ is multiplication by $A$, then the following are equivalent.
(a) A is invertible.
(b) $A \mathbf{x}=\mathbf{0}$ has only the trivial solution.
(c) The reduced row-echelon form of $A$ is $I_{n}$.
(d) A is expressible as a product of elementary matrices.
(e) $A \mathbf{x}=\mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
(f) $A \mathbf{x}=\mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
(g) $\operatorname{det}(A) \neq 0$.
(h) The range of $T_{A}$ is $R^{n}$.
(i) $T_{A}$ is one-to-one.
(j) The column vectors of A are linearly independent.
(k) The row vectors of $A$ are linearly independent.
(l) The column vectors of A span $R^{n}$.
(m) The row vectors of $A$ span $R^{n}$.
(n) The column vectors of $A$ form a basis for $R^{n}$.
(o) The row vectors of $A$ form a basis for $R^{n}$.
( $p$ ) A has rankn.
(q) A has nullity 0 .
(r) The orthogonal complement of the nullspace of $A$ is $R^{n}$.
(s) The orthogonal complement of the row space of $A$ is $\{\mathbf{0}\}$.
(t) $A^{T} A$ is invertible.
(u) $\lambda=0$ is not an eigenvalue of $A$.

This theorem relates all of the major topics we have studied thus far.

## Exercise set

7.1

1. Find the characteristic equations of the following matrices:
(a) $\left[\begin{array}{rr}3 & 0 \\ 8 & -1\end{array}\right]$
(b) $\left[\begin{array}{rr}10 & -9 \\ 4 & -2\end{array}\right]$
(c) $\left[\begin{array}{ll}0 & 3 \\ 4 & 0\end{array}\right]$
(d) $\left[\begin{array}{rr}-2 & -7 \\ 1 & 2\end{array}\right]$
(e) $\left[\begin{array}{ll}0 & 0 \\ 0 & 0\end{array}\right]$
(f) $\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$
2. Find the eigenvalues of the matrices in Exercise 1.
3. Find bases for the eigenspaces of the matrices in Exercise 1.
4. Find the characteristic equations of the following matrices:
(a) $\left[\begin{array}{rrr}4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1\end{array}\right]$
(b) $\left[\begin{array}{rrr}3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2\end{array}\right]$
(c) $\left[\begin{array}{rrr}-2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4\end{array}\right]$
(d) $\left[\begin{array}{rrr}-1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1\end{array}\right]$
(e) $\left[\begin{array}{rrr}5 & 0 & 1 \\ 1 & 1 & 0 \\ -7 & 1 & 0\end{array}\right]$
(f) $\left[\begin{array}{rrr}5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2\end{array}\right]$
5. Find the eigenvalues of the matrices in Exercise 4.
6. Find bases for the eigenspaces of the matrices in Exercise 4.
7. Find the characteristic equations of the following matrices:
(a) $\left[\begin{array}{rrrr}0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1\end{array}\right]$
(b) $\left[\begin{array}{rrrr}10 & -9 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -2 & -7 \\ 0 & 0 & 1 & 2\end{array}\right]$
8. Find the eigenvalues of the matrices in Exercise 7.
9. Find bases for the eigenspaces of the matrices in Exercise 7.
10. By inspection, find the eigenvalues of the following matrices:
(a) $\left[\begin{array}{rr}-1 & 6 \\ 0 & 5\end{array}\right]$
(b) $\left[\begin{array}{rrr}3 & 0 & 0 \\ -2 & 7 & 0 \\ 4 & 8 & 1\end{array}\right]$
(c) $\left[\begin{array}{rrrr}-\frac{1}{3} & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\end{array}\right]$
11. Find the eigenvalues of $A^{9}$ for

$$
A=\left[\begin{array}{rrrr}
1 & 3 & 7 & 11 \\
0 & \frac{1}{2} & 3 & 8 \\
0 & 0 & 0 & 4 \\
0 & 0 & 0 & 2
\end{array}\right]
$$

12. Find the eigenvalues and bases for the eigenspaces of $A^{25}$ for

$$
A=\left[\begin{array}{rrr}
-1 & -2 & -2 \\
1 & 2 & 1 \\
-1 & -1 & 0
\end{array}\right]
$$

13. Let $A$ be a $2 \times 2$ matrix, and call a line through the origin of $R^{2}$ invariant under $A$ if $A \mathbf{x}$ lies on the line when $\mathbf{x}$ does. Find equations for all lines in $R^{2}$, if any, that are invariant under the given matrix.
(a) $\quad A=\left[\begin{array}{rr}4 & -1 \\ 2 & 1\end{array}\right]$
(b) $A=\left[\begin{array}{rr}0 & 1 \\ -1 & 0\end{array}\right]$
(c) $A=\left[\begin{array}{ll}2 & 3 \\ 0 & 2\end{array}\right]$
14. Find $\operatorname{det}(A)$ given that $A$ has $p(\lambda)$ as its characteristic polynomial.
(a) $p(\lambda)=\lambda^{3}-2 \lambda^{2}+\lambda+5$
(b) $p(\lambda)=\lambda^{4}-\lambda^{3}+7$

Hint See the proof of Theorem 7.1.4.
15. Let $A$ be an $n \times n$ matrix.
(a) Prove that the characteristic polynomial of $A$ has degree $n$.
(b) Prove that the coefficient of $\lambda^{n}$ in the characteristic polynomial is 1 .
16. Show that the characteristic equation of a $2 \times 2$ matrix $A$ can be expressed as $\lambda^{2}-\operatorname{tr}(A) \lambda+\operatorname{det}(A)=0$, where $\operatorname{tr}(A)$ is the trace of $A$.
17. Use the result in Exercise 16 to show that if

$$
A=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]
$$

then the solutions of the characteristic equation of $A$ are

$$
\lambda=\frac{1}{2}\left[(a+d) \pm \sqrt{(a-d)^{2}+4 b c}\right]
$$

Use this result to show that $A$ has
(a) two distinct real eigenvalues if $(a-d)^{2}+4 b c>0$
(b) two repeated real eigenvalues if $(a-d)^{2}+4 b c=0$
(c) complex conjugate eigenvalues if $(a-d)^{2}+4 b c<0$
18. Let $A$ be the matrix in Exercise 17. Show that if $(a-d)^{2}+4 b c>0$ and $b \neq 0$, then eigenvectors of $A$ corresponding to the eigenvalues

$$
\lambda_{1}=\frac{1}{2}\left[(a+d)+\sqrt{(a-d)^{2}+4 b c}\right] \quad \text { and } \quad \lambda_{2}=\frac{1}{2}\left[(a+d)-\sqrt{(a-d)^{2}+4 b c}\right]
$$

are

$$
\left[\begin{array}{c}
-b \\
a-\lambda_{1}
\end{array}\right] \quad \text { and } \quad\left[\begin{array}{c}
-b \\
a-\lambda_{2}
\end{array}\right]
$$

respectively.
19. Prove: If $a, b, c$, and $d$ are integers such that $a+b=c+d$, then

$$
A=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]
$$

has integer eigenvalues-namely, $\lambda_{1}=a+b$ and $\lambda_{2}=a-c$.
20. Prove: If $\lambda$ is an eigenvalue of an invertible matrix $A$, and $\mathbf{x}$ is a corresponding eigenvector, then $1 / \lambda$ is an eigenvalue of $A^{-1}$, and $\mathbf{x}$ is a corresponding eigenvector.
21. Prove: If $\lambda$ is an eigenvalue of $A$, $\mathbf{x}$ is a corresponding eigenvector, and $s$ is a scalar, then $\lambda-s$ is an eigenvalue of $A-s I$, and $\mathbf{x}$ is a corresponding eigenvector.
22. Find the eigenvalues and bases for the eigenspaces of

$$
A=\left[\begin{array}{lll}
-2 & 2 & 3 \\
-2 & 3 & 2 \\
-4 & 2 & 5
\end{array}\right]
$$

Then use Exercises 20 and 21 to find the eigenvalues and bases for the eigenspaces of
(a) $A^{-1}$
(b) $A-3 I$
(c) $A+2 I$
23. (a) Prove that if $A$ is a square matrix, then $A$ and $A^{T}$ have the same eigenvalues.

Hint Look at the characteristic equation $\operatorname{det}(\lambda I-A)=0$.
(b) Show that $A$ and $A^{T}$ need not have the same eigenspaces.

Hint Use the result in Exercise 18 to find a $2 \times 2$ matrix for which $A$ and $A^{T}$ have different eigenspaces.

Discussion Discovery
24. Indicate whether each statement is always true or sometimes false. Justify your answer by giving a logical argument or a counterexample. In each part, $A$ is an $n \times n$ matrix.
(a) If $A \mathbf{x}=\lambda \mathbf{x}$ for some nonzero scalar $\lambda$, then $\mathbf{x}$ is an eigenvector of $A$.
(b) If $\lambda$ is not an eigenvalue of $A$, then the linear system $(\lambda I-A) \mathbf{x}=\mathbf{0}$ has only the trivial solution.
(c) If $\lambda=0$ is an eigenvalue of $A$, then $A^{2}$ is singular.
(d) If the characteristic polynomial of $A$ is $p(\lambda)=\lambda^{n}+1$, then $A$ is invertible.
25. Suppose that the characteristic polynomial of some matrix $A$ is found to be $p(\lambda)=$ $(\lambda-1)(\lambda-3)^{2}(\lambda-4)^{3}$. In each part, answer the question and explain your reasoning.
(a) What is the size of $A$ ?
(b) Is $A$ invertible?
(c) How many eigenspaces does $A$ have?
26. The eigenvectors that we have been studying are sometimes called right eigenvectors to distinguish them from left eigenvectors, which are $n \times 1$ column matrices $\mathbf{x}$ that satisfy $\mathbf{x}^{T} A=\mu \mathbf{x}^{T}$ for some scalar $\mu$. What is the relationship, if any, between the right eigenvectors and corresponding eigenvalues $\lambda$ of $A$ and the left eigenvectors and corresponding eigenvalues $\mu$ of $A$ ?

