
Always show (justify) your work unless otherwise stated!

(8) 1. Solve, by Gauss-Jordan elimination, the following system of linear equations.

[illegible]

Solution.

$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 2 & -14 & -14 \end{array}\right] \xrightarrow{(-2)R_2 \text{ add to } R_3} \left[\begin{array}{cccc|c} 1 & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{(2)R_2 \text{ add to } R_1} \left[\begin{array}{cccc|c} 1 & 1 & 0 & -10 & -9 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The last matrix is in RREF. We set $x_2 = s$, $x_4 = t$ (the unknowns that do not correspond to leading 1-s are parameters). Then we solve and get $x_3 = -7 + 7t$ and $x_1 = -9 + 10t - s$. So, the solution is $x_1 = -9 + 10t - s$, $x_2 = s$, $x_3 = -7 + 7t$, $x_4 = t$, where s, t range through $\mathbb{R}$.

(8) 2. In the following system k is a constant.

$$\begin{aligned} 2x + 2ky &= 1 \\ 4x + 8y &= k \end{aligned}$$

(a) Find all values of k, if any, such that the system has a unique solution.
(b) Find all values of k, if any, such that the system is inconsistent.
(c) Find all values of k, if any, such that the system has infinitely many solutions.

Note: you are NOT asked to solve the system. So, do NOT solve it.

Solution (one of many). We first partially row-reduce the augmented matrix:

$\left[\begin{array}{cc:c} 2 & 2k & 1 \\ 4 & 8 & k \end{array}\right] \xrightarrow{(-2)R_1 \text{ add to } R_2} \left[\begin{array}{cc:c} 2 & 2k & 1 \\ 0 & 8-4k & k-2 \end{array}\right]$. We see that if $k=2$ then the second row becomes a zero row and the associated system will have infinitely many solutions. This answers part (c). On the other hand if $k \neq 2$ then we can divide the second row by $8-4k$ to get the following matrix: $\left[\begin{array}{cc:c} 2 & 2k & 1 \\ 0 & 1 & -\frac{1}{4} \end{array}\right]$. Since this matrix can obviously be row-reduced to the identity matrix, it follows that the original system has a unique solution. This settles (a). Consequently, the system is never inconsistent, answering (b).

(7) 3. Let , $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ -2 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. In each of the following cases calculate the defined expressions or write "undefined" beside the undefined expressions.

(a) $A + 2B^T$

$$A + 2B^T = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 4 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & -2 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 0 & -4 \\ -2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -2 \\ -1 & 2 & 4 \end{bmatrix}$$

(b) $C^2 + C^{-1}$

$C^2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. $\det C = -1$, and $C^{-1} = -\begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$.

Hence $C^2 + C^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}$.

(c) $BA+C^T$

Since BA is a 3×3 matrix, and C^T is a 2×2 matrix; this expression is not well defined.

(9) 4. (a) Find A^{-1} if $A=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

(b) Write the following system in a matrix form, then use your answer in part (a) to solve it. (No points will be awarded if other methods are used.)

$$\begin{aligned} x \quad\quad + z &= 1 \\ y + z &= 2 \\ y + 2z &= 3 \end{aligned}$$

Solution (one way; using row reduction is also fine). (a) We compute $\det(A)=1$ and $adj(A)=\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$. Hence $A^{-1}=\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$.

(b) The matrix from of the system is $A\mathbf{x}=\mathbf{b}$, where A is as above, $\mathbf{x}=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $\mathbf{b}=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Multiplying both sides of $A\mathbf{x}=\mathbf{b}$ by A^{-1} to the left gives $\mathbf{x}=A^{-1}\mathbf{b}$. Using what we found in part (a) this means that $\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, which, after multiplying, gives,

$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. So, the unique solution is $x=0$, $y=1$, $z=1$.

(10) 5. We reduce $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ to the identity matrix with the following row-operations.

$$\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(a) Find elementary matrics E_1, E_2 and E_3 such that $E_3E_2E_1A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Solution. The first operation we apply above clearly exchanges the two rows. Applying the same operation to the identiy matrix gives $E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The second operation is $\xrightarrow{(\frac{1}{2})R_2}$, and doing that to I, yields $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$. Finally, the third operation is $\xrightarrow{(-1)R_2 \text{ add to } R_1}$, and so $E_3 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

(b) Find E_1^{-1}, E_2^{-1} and E_3^{-1}.

These correspond to the inverse operations. So, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{exchange } R_2 \text{ and } R_1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = E_1^{-1}$; $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{(2)R_2} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = E_2^{-1}$, and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_2 \text{ add to } R_1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = E_3^{-1}$.

(c) Write A as a product of elementary matrices.

Since $E_3E_2E_1A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, it follows that $A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(9) 6. Evaluate $\det A$ in the following cases.

(a) $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$

Solution. (a) This is a diagonal matrix; so $\det(A) = (1)(2)(3)(4)(5) = 120$

(b) $A = \begin{bmatrix} 1 & 2 & 2 & 2 & 5 \\ 0 & 2 & 3 & 0 & 5 \\ 2 & 0 & 3 & 4 & 5 \\ 3 & 0 & 0 & 6 & 5 \\ 4 & 0 & 0 & 8 & 5 \end{bmatrix}$

Solution. (b) The fourth column is twice the first one. Hence $\det(A) = 0$.

(c) $A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$.

Solution. (c) Expanding aling, say, the second row, gives

$$\det A = (1)(-1)^{1+2} \begin{vmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} + 0 + (1)(-1)^{2+3} \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{vmatrix} + 0 = 2\ .$$

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(9) 7. (a) For $A=\begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$ compute the cofactors C_{12} and C_{33}.

Solution.

$C_{12}=(-1)^{1+2}\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{vmatrix}=2$. $C_{33}=(-1)^{3+3}\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{vmatrix}=0$

(b) Let B be a 3×3 (unknown) matrix such that $B\,adj(B)=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Find $\det(B)$.

Solution (one of many).

It follows from what we are given that $B\,adj(B)=2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, i.e., that $B\left(\frac{1}{2}adjB\right)=I$.

Hence $\frac{1}{2}adjB=B^{-1}$. Since $B^{-1}=\frac{1}{\det B}adj(B)$, it follows that $\det B=2$.