Brief Solutions (or just answers) of the midterm exam questions

1.

1. Solve each system by the method indicated, showing your work (full marks will not be awarded for the answer alone or any other method than the one requested.)

[5] (a) Use *Cramer's Rule*

$$\begin{aligned} 2x_1 &- 3x_2 &= 5 \\ 3x_1 &- 5x_2 &= 7 \end{aligned}$$

Solution. $A = \begin{bmatrix} 2 & -3 \\ 3 & -5 \end{bmatrix}$, $A_1 = \begin{bmatrix} 5 & -3 \\ 7 & -5 \end{bmatrix}$, $A_2 = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$.

$x_1 = \dfrac{\det A_1}{\det A} = 4$, $x_2 = \dfrac{\det A_2}{\det A} = 1$.

1. Solve each system by the method indicated, showing your work (full marks will not be awarded for the answer alone or any other method than the one requested.)

[5] (a) Use *Cramer's Rule*

$$\begin{aligned} 2x_1 &- 3x_2 &= 5 \\ 3x_1 &- 5x_2 &= 7 \end{aligned}$$

Solution. The augmented matrix is $\left[\begin{array}{ccc|c} 2 & -4 & -3 & 2 \\ -1 & 2 & 2 & 2 \end{array}\right]$, and its row reduced echelon form is $\left[\begin{array}{ccc|c} 1 & -2 & 0 & 10 \\ 0 & 0 & 1 & 6 \end{array}\right]$. Solving the associated system gives $x = 10 + 2t$, $y = t$, $z = 6$.

2. Suppose that the augmented matrix of a system in variables x, y and z has been partially reduced through elementary row operations to the following form:

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 0 & a \\ 0 & a & a & b \\ 0 & 0 & 0 & 0 \end{array}\right].$$

[3] (a) Find all values (if any) of a and b for which the system is inconsistent.

Solution. The RREF of the above matrix is $\begin{bmatrix} 1 & 0 & -2 & | & 1 \\ 0 & 1 & 0 & | & a \\ 0 & 0 & a & | & b-a \end{bmatrix}$. From there we see that the system is inconsistent for $a = 0$, $b \neq 0$.

[3] (b) Find all values (if any) of a and b for which the system has a unique solution.

Solution. $a \neq 0$, and any *b*.

[4] (c) Solve the system in the case that there are infinitely many solutions.

Solution. The system has infinitely many solutions when both *a* and *b* are 0. In that case we find that $x = 1 + 2t$, $y = 0$, $z = t$ is the general solution (*t* ranges through all numbers).

3. Suppose that $2A^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$. Find:

[4] (a) A

Solution. $A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$, and so $A = \left(\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\right)^{-1} = 2\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}^{-1}$. Now find the inverse of the matrix $\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$ and multiply it by two to get that $A = \begin{bmatrix} \frac{4}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$.

[2] (b) A^T

Solution. $A^T = \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}$

[3] (c) adj(A).

Solution. $adj(A) = \det(A)A^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{4}{3} \end{bmatrix}$.

[10] 4. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} -2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$. Determine whether each expression is defined. If it is defined, calculate the resulting matrix.

(a) AB

Solution. *AB* is undefined.

(b) CA^T

Solution. $CA^T = \begin{bmatrix} -2 & 2 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}$.

(c) B^2A

Solution. $B^2 A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$

(d) $A + C$.

Solution. Undefined.

[8] 5. Use row operations to find the inverse of $A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -\frac{1}{2} & 1 \\ 0 & -1 & 3 \end{bmatrix}$.

Answer: $A^{-1} = \begin{bmatrix} -1 & -2 & 1 \\ 0 & -6 & -2 \\ 0 & -2 & 1 \end{bmatrix}$.

6. Calculate determinants as instructed.

(a) Given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$, deduce the value of each of the following (marks for answer only):

[3] i. $\begin{vmatrix} a+2d & b+2e & c+2f \\ g & h & i \\ d & e & f \end{vmatrix}$

Answer: -7

[3] ii. $\begin{vmatrix} 3a-b & b & c \\ 3d-e & e & f \\ 3g-h & h & i \end{vmatrix}$

Answer: 21

[7] (b) Use row operations, cofactor expansion or any combination of these (show work) to find:

$$\begin{vmatrix} 1 & 0 & 1 & -1 \\ -2 & 0 & 1 & 0 \\ 2 & 3 & 1 & 1 \\ 1 & 0 & 2 & 1 \end{vmatrix}$$

Answer: -24.