Question 19, page 190 text.

(a) If $det(A) \pi 0$ then $det(AdjA) \pi 0$ (proven in class)

(b) Proving that $det(AdjA) \pi 0$ implies $det(A) \pi 0$.

Assume $det(AdjA) \pi 0$. We now show that under that assumption we could not have $det(A) = 0$. So, suppose $det(A) = 0$. If A is the zero matrix, then Adj(A) is also the zero matrix and we get a contradiction. So, assume A is not the zero matrix. Since $det(A) = 0$ (after using “expansion along the first row of A” for $det(A)$, we have:

$a_{11}(\ 1)^{1+1} det(A_{11}) + a_{12}(\ 1)^{1+2} det(A_{12}) + \dots + a_{1n}(\ 1)^{1+n} det(A_{1n}) = 0$.

Now replace the second row of A with the first row of A to get a new matrix B(2) (just replace – no interchanging of rows). This new matrix has the first row equal to the second row; so its determinant is also 0. We expand along the second row to conclude that

$a_{11}(\ 1)^{2+1} det(A_{21}) + a_{12}(\ 1)^{2+2} det(A_{22}) + \dots + a_{1n}(\ 1)^{2+n} det(A_{2n}) = 0$

(Note that I’ve been using the terms in the second row of B(2); these are not the same as the entries in the second row in A. However, the associated cofactors of A and B(2) are the same since A and B(2) differ only in the second row which is the one that is deleted to get the cofactors.)

Now start again with A and replace the third row with the first row to get a new matrix B(3). Since B(3) has two equal rows its determinant is 0. Expanding along the third row we get

$a_{11}(\ 1)^{3+1} det(A_{31}) + a_{12}(\ 1)^{3+2} det(A_{32}) + \dots + a_{1n}(\ 1)^{3+n} det(A_{3n}) = 0$.

Keep dong this to get B(4), B(5), …, B(n). The last step (B(n)) will yield

$a_{11}(\ 1)^{n+1} det(A_{n1}) + a_{12}(\ 1)^{n+2} det(A_{n2}) + \dots + a_{1n}(\ 1)^{n+n} det(A_{nn}) = 0$.

Let’s gather all these in the following set of equations (call it (*)):

$$
\begin{aligned}
&a_{11}(\ 1)^{1+1} det(A_{11}) + a_{12}(\ 1)^{1+2} det(A_{12}) + \dots + a_{1n}(\ 1)^{1+n} det(A_{1n}) = 0\\
&a_{11}(\ 1)^{2+1} det(A_{21}) + a_{12}(\ 1)^{2+2} det(A_{22}) + \dots + a_{1n}(\ 1)^{2+n} det(A_{2n}) = 0\\
&a_{11}(\ 1)^{3+1} det(A_{31}) + a_{12}(\ 1)^{3+2} det(A_{32}) + \dots + a_{1n}(\ 1)^{3+n} det(A_{3n}) = 0\\
&\vdots\\
&a_{11}(\ 1)^{n+1} det(A_{n1}) + a_{12}(\ 1)^{n+2} det(A_{n2}) + \dots + a_{1n}(\ 1)^{n+n} det(A_{nn}) = 0.
\end{aligned}
$$

Let us looked at this from another perspective. Consider the system

$$
\begin{aligned}
&x_1(\ 1)^{1+1} det(A_{11}) + x_2(\ 1)^{1+2} det(A_{12}) + \dots + x_n(\ 1)^{1+n} det(A_{1n}) = 0\\
&x_1(\ 1)^{2+1} det(A_{21}) + x_2(\ 1)^{2+2} det(A_{22}) + \dots + x_n(\ 1)^{2+n} det(A_{2n}) = 0\\
&x_1(\ 1)^{3+1} det(A_{31}) + x_2(\ 1)^{3+2} det(A_{32}) + \dots + x_n(\ 1)^{3+n} det(A_{3n}) = 0\\
&\vdots\\
&x_1(\ 1)^{n+1} det(A_{n1}) + x_2(\ 1)^{n+2} det(A_{n2}) + \dots + x_n(\ 1)^{n+n} det(A_{nn}) = 0.
\end{aligned}
$$

It is a homogeneous system and the coefficient matrix of that system is apparently Adj(A). Since A is not the zero matrix, (*) tells us that the system has a non-zero solution !. This could happen only of the coefficient matrix is not invertible, i.e., if has a determinant equal to 0 (theory of systems). So Adj(A)=0 (contradiction).