Midterm **Brief** Solutions

1. Use the Principle of Mathematical Induction to prove that,
for all integers $n \geq 1$:
1 + 5 + 9 + ... + (4n -3) = $2n^2 - n$:
(Note: no partial credit will be given if you do not use the PMI.)

Solution. This is true for $n = 1$.
Assume it is true that $1 + 5 + \ldots + (4k - 3) = 2k^2 - k$.
We now show, under that assumption, that
$1 + 5 + \ldots + (4k - 3) + (4k + 1) = 2(k + 1)^2 - (k + 1)$.
Using the inductive assumption, we see that the LFH is $2k^2 - k + (4k + 1)$; and the RHS is $2k^2 + 4k + 2 - k - 1$. It is easy to see that these two are equal.

2. (value 8) Identities:

$$\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1) ; \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Use the above identities to evaluate the sums of:

(a) $\sum_{k=1}^{100} k^2(k-3)$

(b) $10^3 + 11^3 + 12^3 + \ldots + 100^3$

Solution (a). $\sum_{k=1}^{100} k^2(k-3) = \sum_{k=1}^{100} k^3 - 3\sum_{k=1}^{100} k^2 = \frac{100^2(101)^2}{4} - 3\frac{1}{6}100(101)(201) = 24487450$

(b) $10^3 + 11^3 + 12^3 + \ldots + 100^3 = \sum_{k=1}^{100} k^3 - \sum_{k=1}^{9} k^3 = \frac{100^2(101)^2}{4} - \frac{9^2 10^2}{4} = 25500475$

3. (value 12) Consider the sequence $\{c_n\}$ defined by $c_1 = \frac{1}{2}$, $c_{n+1} = c_n^2$
for integers $n \geq 1$.:
(a) Show that $\{c_n\}$ is monotonic.
(b) Show that $\{c_n\}$ is bounded.
(c) Does it follow from (a) and (b) that $\{c_n\}$ converges? Briefly explain in a sentence or two.
(d) Find the limit of $\{c_n\}$ If the limit exists.

Solution. (a) $c_2 = \frac{1}{4}$, and obviously $c_1 > c_2$. Assume $c_k > c_{k+1}$. Then $c_k^2 > c_{k+1}^2$ and so $c_{k+1} > c_{k+2}$. It follows by math induction that the sequence decreases all the time.

(b) Since the sequence decreases, it is bounded from above by $c_1 = \frac{1}{2}$. Since the members of the sequence are all squares of numbers, it follows that they are bounded from below by 0.

(c) Every bounded monotonic sequence converges (Theorem). So this one too.

(d). Now that we know that $\lim_{n\to\infty} c_n$ exists, we may write $\lim_{n\to\infty} c_n = L$. It follows from $c_{n+1} = c_n^2$ that $L = L^2$. Solving this gives $L = 0$ or $L = 1$. The latter is certainly not the limit (since all the members of the sequence are less than $\frac{1}{2}$); hence $L = 0$.

4. (value 10) (a) Find an explicit formula for the nth term of the recursive sequence: $c_1 = 0$, $c_{n+1} = \frac{-1}{3}c_n + 2$

(b) Find the limit of the sequence, if it exists.

Hint: recall that for a first-order linear difference equation: (given for free in the test; not shown here).

Solution. (a) In the formula that is given we substitute $a = 0$, $d = 2$ and $r = \frac{-1}{3}$, to get that $c_n = 2\left[\frac{\left(\frac{-1}{3}\right)^{n-1} - 1}{-\frac{1}{3} - 1}\right] = \frac{-3}{2}\left[\left(\frac{-1}{3}\right)^{n-1} - 1\right]$.

(b) Since $-1 < r < 1$ the limit exists. Since $\lim_{n\to\infty}\left(\frac{-1}{3}\right)^{n-1} = 0$, we have that $\lim_{n\to\infty} c_n = \frac{3}{2}$

5. (value 10) Let v = 2+3i and w = 2+i be complex numbers. Calculate each of the following and express in the form z = a + bi:

(a) v + w

(b) wv

(c) $\frac{v}{w}$

Answers. (a) $v + w = 4 + 4i$ (b) $wv = -2 + 8i$, (c) $\frac{v}{w} = \frac{7}{5} + \frac{4}{5}i$

6. (value 10) Let $P_3(x) = x^3 + 6x^2 + 9x + 4$.
(a) List all possible zeroes of $P_3(x)$ by the Rational Root Theorem;
(b) Determine the number of positive and negative zeroes of $P_3(x)$ by the Dercartes's Rule of Sign;
(c) Find all zeroes of $P_3(x)$

(a) Possible rational zeros are ± 4, ± 2, ± 1.
(b) $P_3(x)$ could have at most 3 zeros. By the Dercartes's Rule of Signs, there are no positive real solution, and there are 3 or 1 negative real solutions.
(c) We check the negative numbers obtained in (a) and we see that -1 and -4 are solutions. Dividing $P_3(x)$ by $x + 4$ (long division) gives $(x+1)^2$, so -1 is a zero of multiplicity 2.