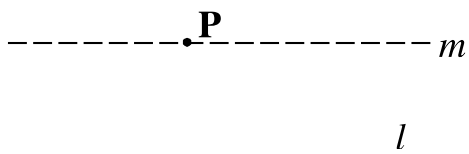
HYPERBOLIC GEOMETRY

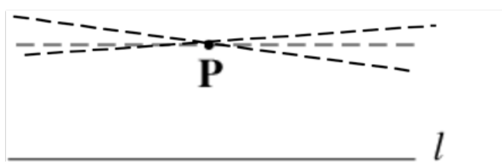
Recall the fifth axiom of the Euclidean geometry:

5. For every line *l* and every point P that does not lie on *l*, there exists a unique line *m* through P and parallel to *l*.

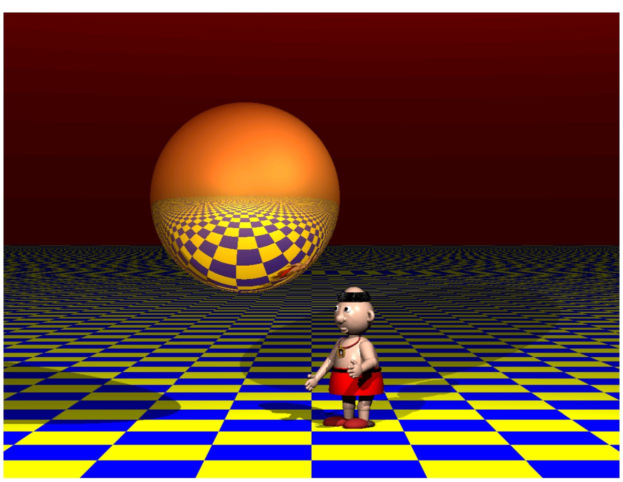


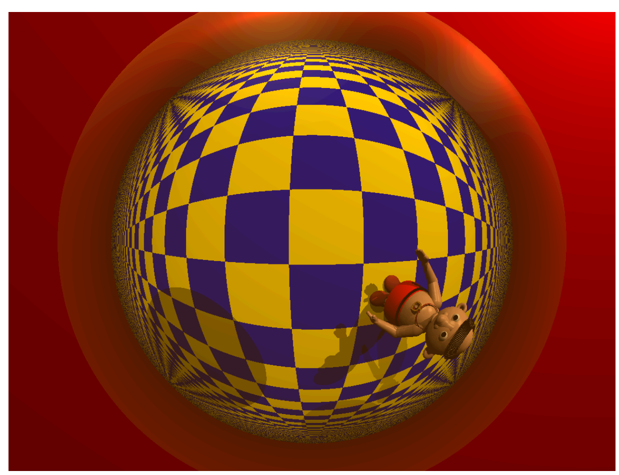
In hyperbolic geometry this axiom is replaced by

5. For every line *l* and every point P that does not lie on *l*, there exist infinitely many lines through P that are parallel to *l*.



New geometry models immerge, sharing some features (say, curved lines) with the image on the surface of the crystal ball of the surrounding three-dimensional scene.

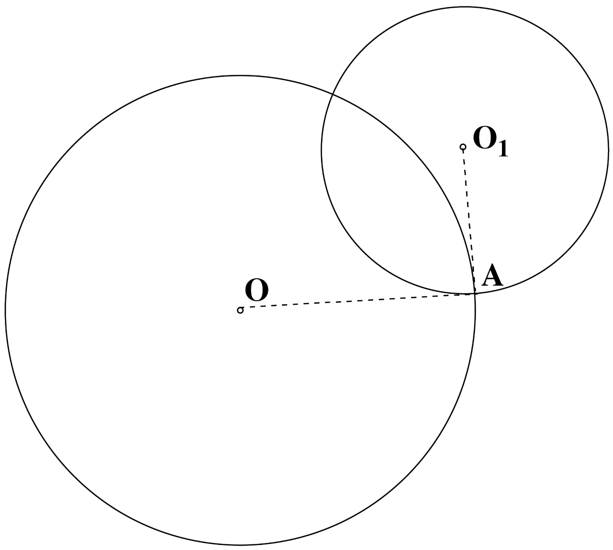




PRELIMINARIES

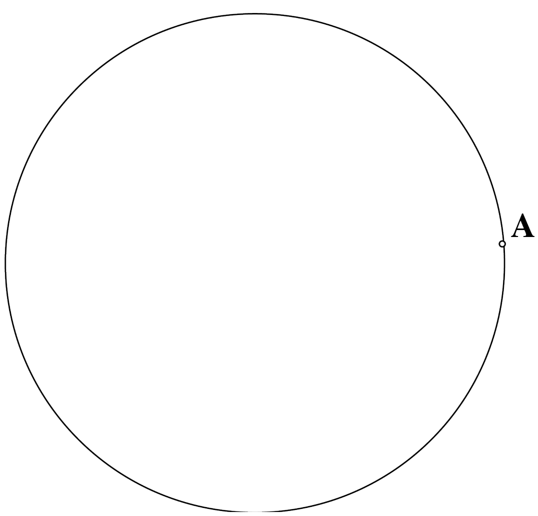
Q. When do two circles intersect at the right angle?

A. Their radii to any the two intersecting points should be perpendicular.

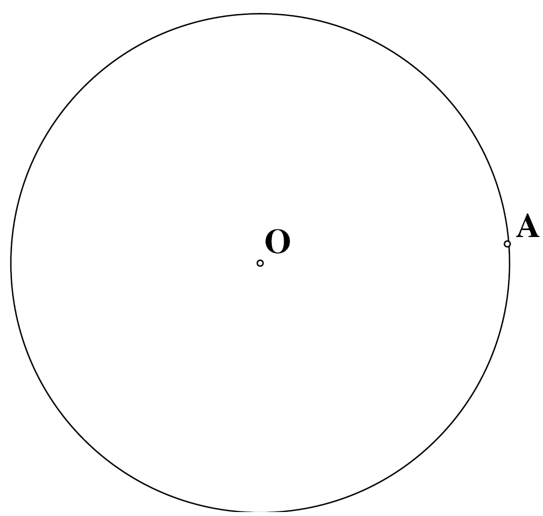


The radial segments *OA* and AO_1 are perpendicular.

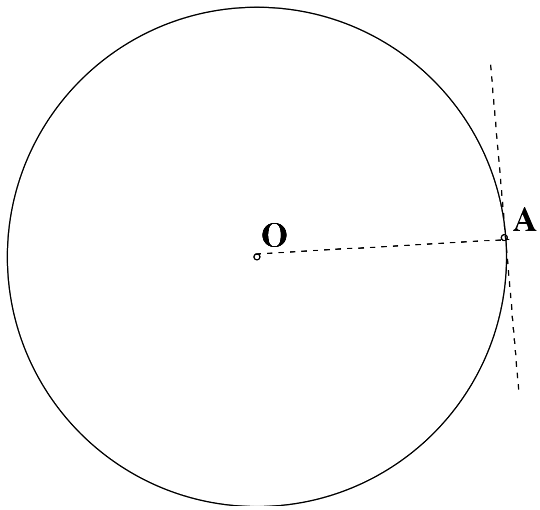
Basic construction #0: given a circle C and a point A on it, construct a bunch of circles intersecting C at A and perpendicular to C.

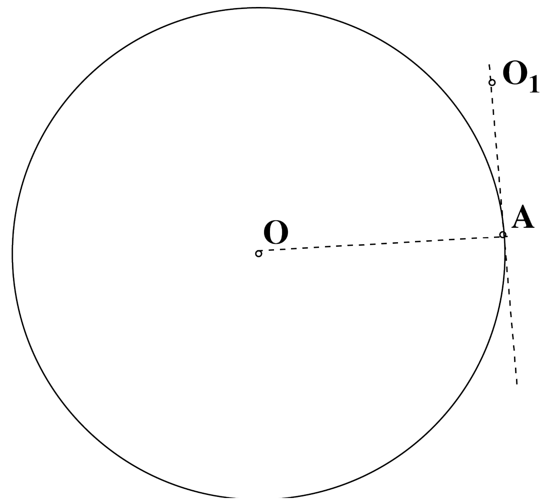


We are given a circle and a point A.



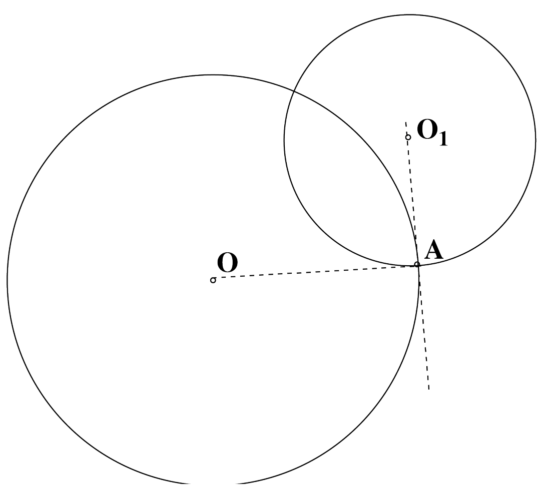
Construct its center O (old construction).

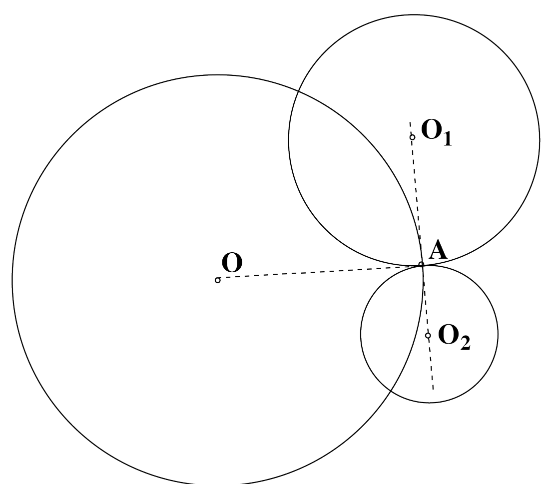




Join OA and construct its perpendicular through A.

Choose any point O1 (other than A) on the perpendicular.

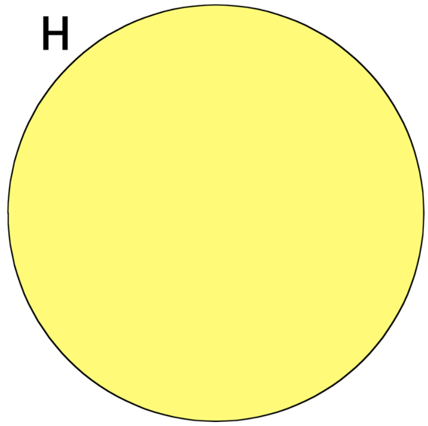




Draw the circle centered at O1 and through A.

Choose O2 and get another perpendicular circle.

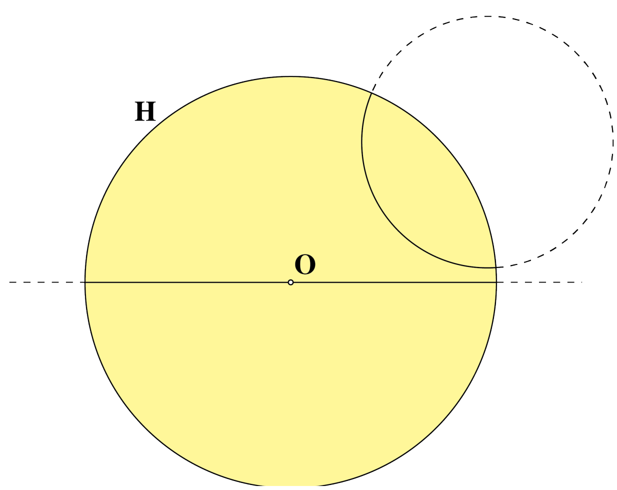
POINCARE MODEL OF HYPERBOLIC GEOMETRY



The space (the new world) consists of all of the points in the interior of a circle. The circle itself is not a part of the space and it plays the role of a horizon.

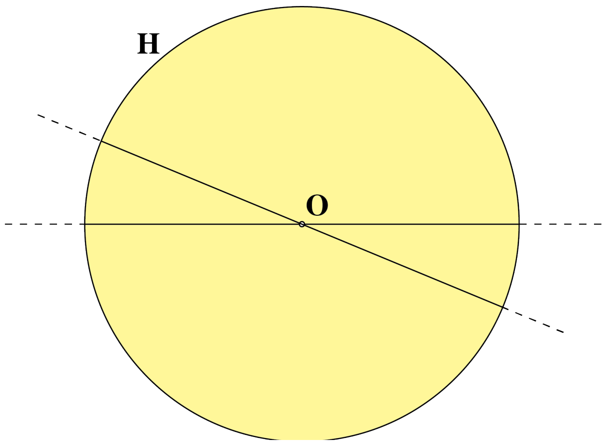
There are two types of lines:
Hyperbolic lines (*h*-lines) of type 1: diameters of the circle *H* without the end-points.

Hyperbolic lines (*h*-lines) of type 2: circles intersecting *H* at the right angle; the part of that circle within *H*.

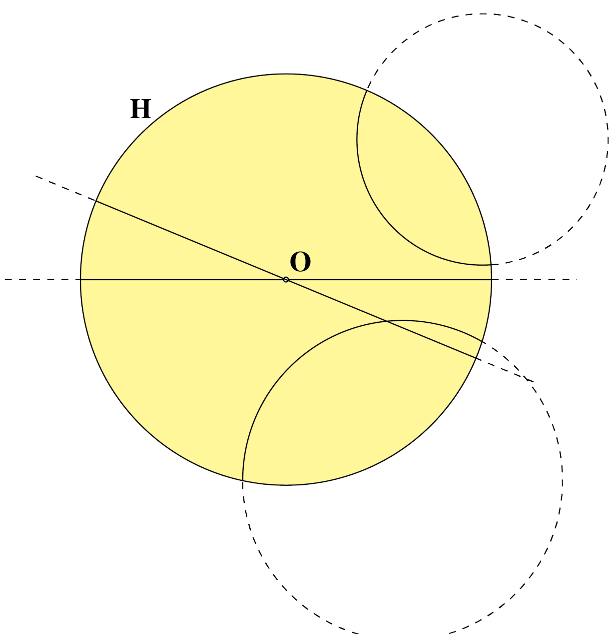


The two hyperbolic lines, one diameter in one arc, are in full stroke.

Basic construction #1. Construct two *h*-lines of type 1, and two *h*-lines of type 2.



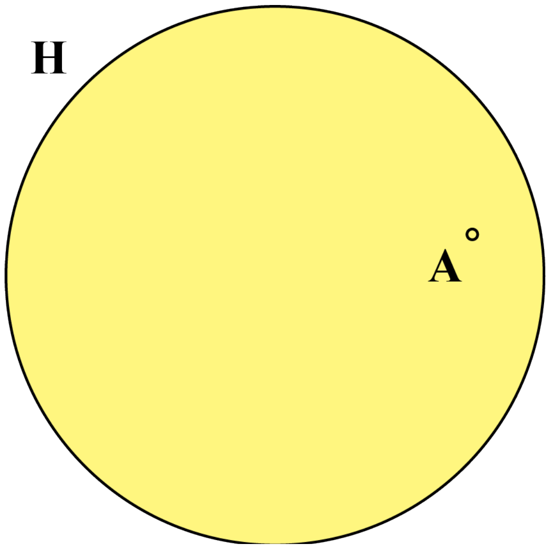
Construct the center O of H; draw any two diameters.



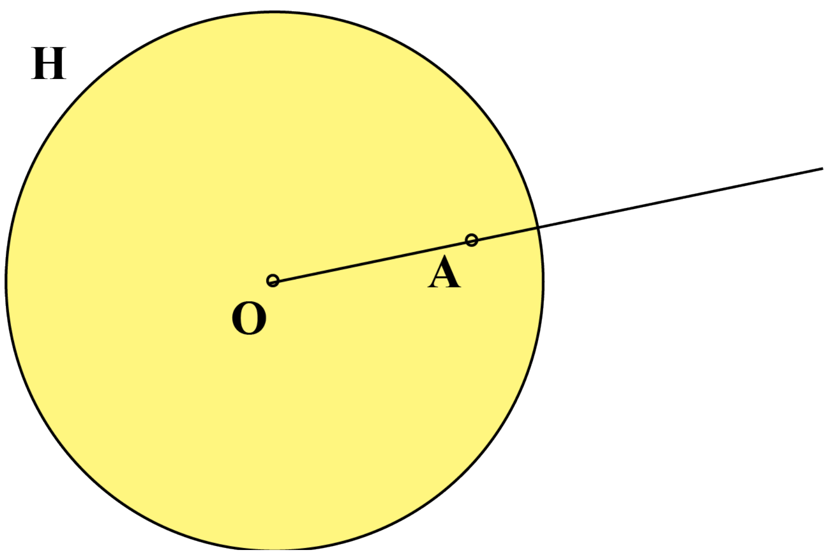
Use construction #0 (above) to get any two circles intersecting H at right angles; the parts of these circles within H are h-lines of type 2.

The notion of being parallel, in hyperbolic or Euclidean plane geometry, is the same as the notion of non-intersecting (**parallel = non-intersecting**). Thus, the top h-line of type 2 is parallel to the other 3 h-lines in the figure. (See Figure 4.3.3 in the textbook and the surrounding text.)

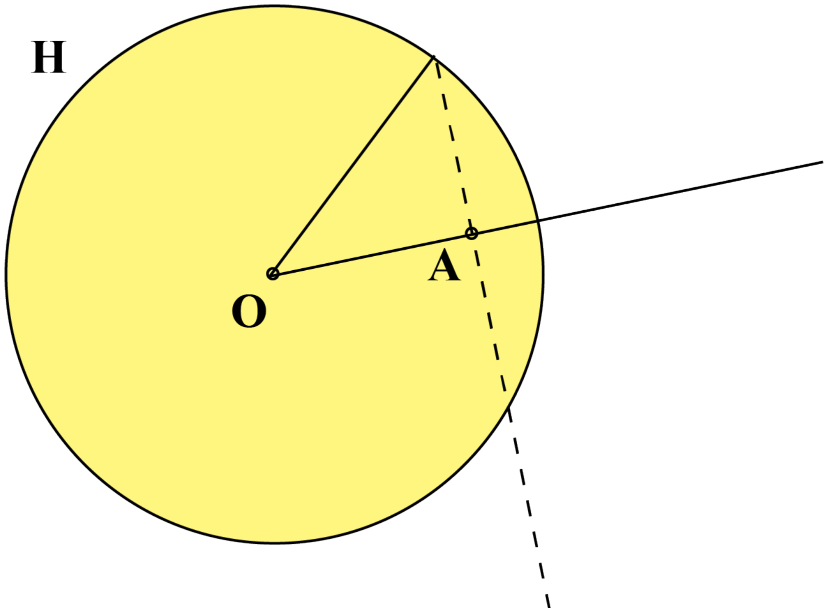
Basic important construction #2. Given a point A, construct one *h*-line of type 2 passing through A.



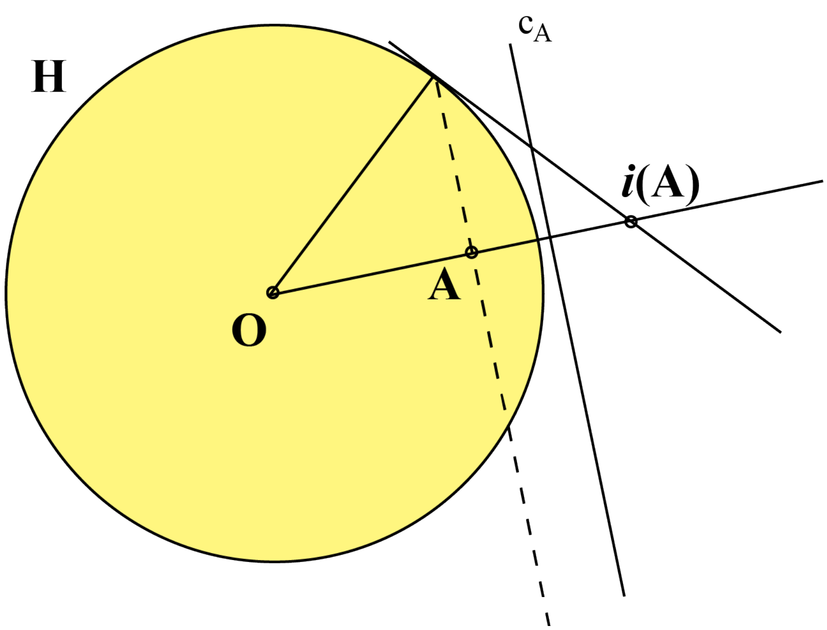
Initially we are given a point A within H.



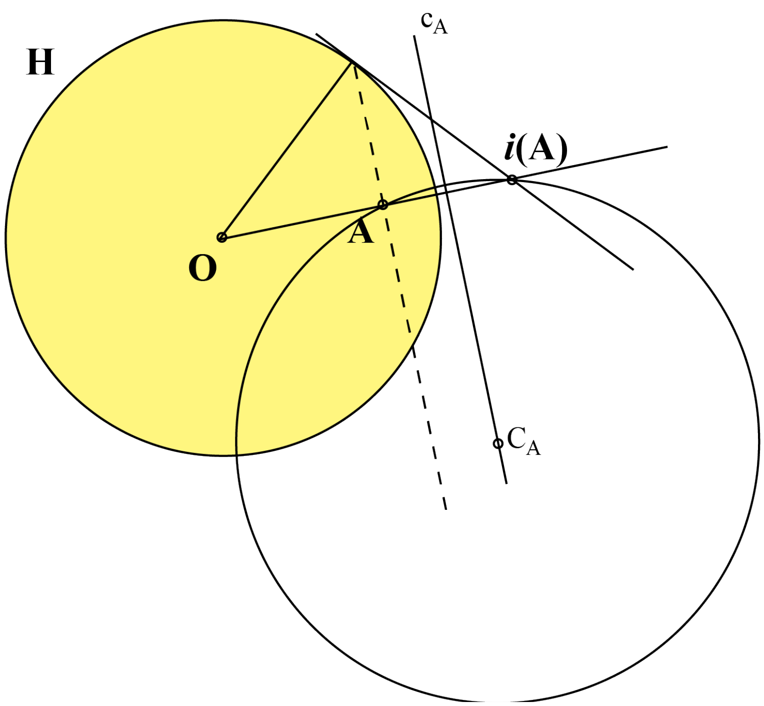
Construct the center O of H, and join it with A.
(The diameter through A is an h-line of type 1 passing through A; we are after an h-line of type 2 through A.)



Construct the perpendicular to OA at A; connect with O one of the two intersection points of that perpendicular with H; get one radius of H.



Construct the perpendicular to that radius is at the end point. This perpendicular intersects the line through OA at i(A). Bisect the line segment A i(A) to get the line c_A.



Choose any point on the line c_A (we chose C_A) and draw a circle center at that point and passing through A; the part of that circle within H is a desired h-line type 2.

Note: the **center-line** c_A contains the centers of <u>all</u> *h*-lines passing through A.