Office Hour. Questions Please:

Textbook 4.4.16:
Problem 16.
Find $c$ such that $f(2) - f(0) = f'(c)(2 - 0)$, $f(x) = \cos(2\pi x), 0 < c < 2$

\[ f(2) - f(0) = \cos(4\pi) - \cos(0) = 0 \Rightarrow f'(c) = 0 \]

\[ f'(x) = (-\sin(2\pi x)) \cdot (2\pi x)' = -2\pi \sin(2\pi x) \]

\[ f'(c) = 0 \Rightarrow \sin(2\pi c) = 0 \Rightarrow 2\pi c = k\pi, \Rightarrow c = \frac{1}{2}k, k = 0, \pm1, \pm2,... \]

(Recall: $\sin \alpha = 0$ if and only if $\alpha = k\pi, k = 0, \pm1, \pm2,...$)
To make $c \in (0,2)$, the only possible choice of $k$ is $k = 1, 2, 3$.

Answer: $c = \frac{1}{2}, 1, \frac{3}{2}$.

Textbook 4.4.38
Not required. I did not explain due to a lack of time.

Textbook 4.3.36:
\[ y = |x + 1| + |x - 1| \quad x \in [-3, 2] \]
\[ y = \begin{cases} 
  x + 1 + x - 1 & x > 1 \\
  x + 1 - x + 1 & x \in [-1,1] \\
  -x - 1 - x + 1 & x < -1 
\end{cases} = \begin{cases} 
  2x & x > 1 \\
  2 & x \in [-1,1] \\
  -2x & x < -1 
\end{cases} \]

\[ \frac{dy}{dx} = \begin{cases} 
  2 & x > 1 \\
  DNE & x = 1 \\
  0 & -1 < x < 1 \\
  DNE & x = -1 \\
  -2 & x < -1 
\end{cases} \]

The derivative does not exist at $x = 1$ and $x = -1$ because the left-hand limit and the right-hand limit are different.
Critical numbers is any number in $[-1,1]$.
For any critical number $c$, $f(c) = 2$
For the endpoint, we have $f(-3) = -2 \cdot (-3) = 6, f(2) = 2 \cdot 2 = 4$
\[ f(-3) = 6 \text{ is the absolute maximum.} \]
\[ f(c) = 2 \text{ is the absolute minimum, } c \in [-1,1]. \]
Textbook 4.3.38.

\[ y = \sin x + \cos x, x \in [0, 2\pi]. \]

\( y \) is a continuous function. So we apply the closed interval method.

\[ y' = \cos x - \sin x \text{ exists everywhere.} \]

\[ y' = 0 \Rightarrow \cos x = \sin x \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4} \]

For the critical numbers, \( x = \frac{\pi}{4}, y = \sin \frac{\pi}{4} + \cos \frac{\pi}{4} = \sqrt{2} \)

\[ x = \frac{5\pi}{4}, y = \sin \frac{5\pi}{4} + \cos \frac{5\pi}{4} = -\sqrt{2} \]

For the endpoints, \( x = 0, y = \sin 0 + \cos 0 = 1 \)

\[ x = 2\pi, y = \sin 2\pi + \cos 2\pi = 1 \]

\[ y = \sin x + \cos x \text{ takes absolute maximum when } x = \frac{\pi}{4}, \text{ the maximum value is } \sqrt{2} \]

\[ y = \sin x + \cos x \text{ takes absolute minimum when } x = \frac{5\pi}{4}, \text{ the minimum value is } -\sqrt{2} \]

Problem Worksheet #8, Question 3:

Find the absolute maximum and minimum of the following functions on the given interval.

Make sure to justify the conditions of the extreme value theorem

(a) \( f(x) = e^{-x} - e^{-2x} \quad x \in [0, 1] \)

\( f(x) \) is continuous. We apply the close interval method.

\[ f'(x) = -e^{-x} + 2e^{-2x} \text{ exists everywhere.} \]

\[ f'(c) = 0 \Rightarrow -e^{-c} + 2e^{-2c} = 0 \]

\[ \Rightarrow -\frac{1}{e^c} + \frac{2}{e^{2c}} = -\frac{e^c}{e^c e^c} + \frac{2}{e^{2c}} = -e^c + 2e^{-2c} = 0 \Rightarrow -e^c + 2 = 0 \Rightarrow e^c = 2 \Rightarrow c = \ln 2 \]

Recall: \( a^b = b \Leftrightarrow x = \log_a b \)

\[ f(0) = 1 - 1 = 0 \]

\[ f(\ln 2) = e^{-\ln 2} - e^{-2\ln 2} = e^{\ln 2^{-1}} - e^{\ln 2^{-2}} = 2^{-1} - 2^{-2} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \]

\[ f(1) = e^{-1} - e^{-2}. \]

\[ f''(x) = e^{-x} - 4e^{-2x} \]

\[ f''(\ln 2) = e^{-\ln 2} - 4e^{-2\ln 2} = \frac{1}{2} - 1 < 0. \text{ Second derivative test says that } f(\ln 2) \text{ is a local maximum.} \]

\[ f(\ln 2) = \frac{1}{4} \text{ is the absolute maximum.} \]

\[ f(0) = 0 \text{ is the absolute minimum.} \]
(b) $f(x) = (x^2 - 1)^3$, $x \in [-1, 2]$

$f$ is continuous. We apply closed interval method.

$f'(x) = 3(x^2 - 1)^2 \cdot (x^2 - 1)' = 3(x^2 - 1)^2 \cdot 2x$

$= 3(x - 1)^2(x + 1)^2 \cdot 2x$ exists everywhere.

$f'(x) = 0 \Rightarrow 3(x - 1)^2(x + 1)^2 \cdot 2x = 0 \Rightarrow x = -1, 0, 1.$

$f(-1) = 0$

$f(0) = (-1)^3 = -1$

$f(1) = 0$

$f(2) = (4 - 1)^3 = 27.$

$f(0) = -1$ is the absolute minimum.

$f(2) = 27$ is the absolute maximum.

(c) $h(x) = x - \ln x$, $x \in [1/2, 2]$

$h$ is continuous. We apply closed interval method.

$h'(x) = 1 - \frac{1}{x}$ does not exist when $x = 0$ (out of the domain).

$h'(x) = 0 \Rightarrow 1 - \frac{1}{x} = 0 \Rightarrow x = 1$

$h(1/2) = \frac{1}{2} - \ln \frac{1}{2} = \frac{1}{2} + \ln 2$

$h(1) = 1 - \ln 1 = 1$

$h(2) = 2 - \ln 2$

Note: Consider $\frac{\ln 2}{1/2} = 2 \ln 2 = \ln 4 > \ln e = 2 \Rightarrow \ln 2 > 1/2$

$h(1/2) > 1$

Note: Since $2 < e, \ln 2 < \ln e = 1 \Rightarrow 2 - \ln 2 > 1$

$h(2) > 1$

Note: $2 - \ln 2 - \left(\frac{1}{2} + \ln 2\right) = \frac{3}{2} - 2 \ln 2 = \frac{3}{2} - \ln 4 = \frac{1}{2}(3 - 2 \ln 4) = \frac{1}{2}(\ln e^3 - \ln 16)$

$e^3 > 2.7^3 > 19 > 16$

$h(2) > h(1/2) > h(1)$

$h(2) = 2 - \ln 2$ is the absolute maximum,

$h(1) = 1$ is the absolute minimum.