ON THE SIGN PATTERNS OF THE SMALLEST SIGNLESS LAPLACIAN EIGENVECTOR

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Abstract. Let H be a connected bipartite graph, whose signless Laplacian matrix is $Q(H)$. Suppose that the bipartition of H is (S,T) and that x is the eigenvector of the smallest eigenvalue of $Q(H)$. It is well-known that x is positive and constant on S, and negative and constant on T.

The resilience of this sign pattern under addition of edges into the subgraph induced by either S or T is investigated and a number of cases in which the sign pattern of x persists are described.

Key words. Signless Laplacian matrix, Eigenvector signs, Bipartite graph, Maximal independent set.

AMS subject classifications. 05C50,15A18,15B48.

1. Introduction. We deal with the signless Laplacian matrix of a graph G. Subject to benign neglect up to very recent times, it has received a lot of attention lately, some of the results of which are summarized in the surveys [3, 4, 5, 6].

Let G be a graph with adjacency matrix $A(G)$ and let $D(G)$ be the diagonal matrix of the vertex degrees of G. The Laplacian matrix of G is $L(G) = D(G) - A(G)$ and the *signless Laplacian matrix* of G is $Q(G) = D(G) + A(G)$. One of the alluring properties of the signless Laplacian matrix is that it is attuned to the bipartiteness of a graph. It is well-known that 0 is an eigenvalue of $Q(G)$ with multiplicity equal to the number of bipartite connected components of G.

Desai and Rao [9] first tackled the issue of what structural bipartiteness properties can be derived from the fact that the smallest eigenvalue $\mu(G)$ of $Q(G)$ is small but non-zero. They showed in a precise sense that a low value of $\mu(G)$ indicates the presence of a nearly-bipartite subgraph of G that is weakly connected to the rest of a graph. Very recently their results were improved in [10]. However, neither [9] nor [10] gives a way to detect such a subgraph in a given graph G.

A constructive approach was taken in [13] where a condition was established for a given subset for $S \subseteq V(G)$ to induce a bipartite subgraph, based on $\mu(G)$ and the Rayleigh quotient for $Q(G)$ of a certain indicator vector of S.

In this note we consider the relationship between bipartiteness and the signless Laplacian from a slightly different angle, studying the sign pattern of an eigenvector that corresponds to $\mu(G)$.

A few words about notation: if $V(G)$ is labelled as $\{1, 2, \ldots, n\}$ and $x \in \mathbb{R}^n$, then for any nonempty subset $S \subseteq V(G)$ we shall mean by $x(S)$ the vector in $\mathbb{R}^{|S|}$ formed by deleting from x all entries not corresponding to elements of S. The all-ones vector of length n will be denoted by $\mathbf{1}_n$ or just $\mathbf{1}$ if the length is clear from the context. We write $v > 0$ to indicate that all the entries of a vector v are strictly positive.

The following fact is well-known:

PROPOSITION 1.1. *Let H be a connected biparite graph with bipartition (S,T). For any eigenvector x corresponding $\mu(H)$ there are is a nonzero number c so that:*

$$x(S) = c\mathbf{1}_{|S|}, x(T) = -c\mathbf{1}_{|T|}.$$

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†Research supported in part by the Science Foundation Ireland under Grant No. SFI/07/SK/I1216b.

Roth [20] has obtained an interesting generalization:

PROPOSITION 1.2. *Let H be a connected biparite graph with bipartition (S, T). Let D be any diagonal matrix and let x be an eigenvector corresponding to the smallest eigenvalue of $Q(H) + D$. Then:*

$$x(S) > 0, x(T) < 0, \tag{1.1}$$

or vice versa.

We are interested in generalizing Proposition 1.1 in a different way, showing that (1.1) continues to hold even when edges are added on one of the sides of H. We keep $D = 0$, however. Let us make the following definition:

DEFINITION 1.3. *Let H be a connected graph and let $S \subseteq V(H)$ be a maximal independent set. We say that H is* S-Roth *if for every eigenvector x corresponding to $\mu(H)$ we have that*

$$x(S) > 0, x(V(H) - S) < 0,$$

or vice versa.

The assumption that S is a maximal independent set is made in order to rule out the uninteresting case when H is bipartite and S is a proper subset of one of the partite sets associated with H. In that case there is a smallest eigenvector of $Q(H)$ that is positive on S but has mixed signs on the complement of S.

REMARK 1.4. *Notice that if H is S-Roth, then $\mu(H)$ must be a simple eigenvalue.*

Proposition 1.1 can now be stated as:

PROPOSITION 1.5. *Let H be a connected biparite graph with bipartition (S, T). Then H is S-Roth.*

In the rest of the paper we shall prove that various classes of graphs are S-Roth. For instance, we show in Corollary 5.5 that any H that is a join of the independent set S and another graph T is S-Roth, provided only that $|S| \geq |T|$.

2. Some useful terms and facts. Terms used without explanation may be found in the book [2]. The minimum and maximum degrees of the graph G will be denoted by $\delta(G)$ and $\Delta(G)$, respectively. The degree of a vertex v in a graph G will be denoted by $d_G(v)$. To indicate that vertices $i, j \in V(G)$ are adjacent we will employ the notation $ij \in E(G)$ or $i \sim_G j$. A cycle (path) on n vertices will be denoted C_n (P_n).

The *disjoint union* of two graphs G_1,G_2 will be denoted by $G_1 \cup G_2$, and is the graph whose vertex and edge sets are the disjoint unions of those of G_1 and G_2.

The *join*, denoted by $G_1 \vee G_2$, is obtained from $G_1 \cup G_2$ by adding to it all edges between vertices in $V(G_1)$ and vertices in $V(G_2)$. Finally, the *complement* $\overline{G}$ of a graph G is the graph on the same vertex set whose edges are those and only those not present in G.

From these definitions the following simple but very useful fact immediately arises:

PROPOSITION 2.1. *Let G be any graph. Then G can be written as a join $G = G_1 \vee G_2$ if and only if $\overline{G}$ is disconnected.*

In light of Proposition 2.1 we can speak of a *maximal join decomposition* of a graph G as $G = G_1 \vee G_2 \vee \ldots \vee G_k$, where each $\overline{G_i}$ is connected.

We also need some facts about the (usual) Laplacian eigenvalues. The first lemma is an expanded statement of [17, Proposition 2.3], including some properties established in the proof.

LEMMA 2.2. *Let G be a graph on n vertices. Then:*

- $\mathbf{1}_n$ *is a zero eigenvector of* $L(G)$.
- *If* G *is connected, then all other eigenvectors of* $L(G)$ *are orthogonal to* $\mathbf{1}_n$.
- *Every zero eigenvector of* $L(G)$ *takes a constant value on each connected component of* G.

LEMMA 2.3. *[15, Theorem 2.1] If* H *is a graph on* n *vertices and* $H = G_1 \vee G_2$*, then* $\lambda_n(L(H)) = n$.

LEMMA 2.4. *Let* $H = G_1 \vee G_2 \vee \ldots \vee G_k$ *be a maximal join decomposition of* H *and assume that* $k \geq 2$*. Let* $n = |V(G)|$*. Let* x *be an eigenvector of* $L(H)$ *corresponding to* n*. Then* x *is constant on the vertex set of each* G_i*,* $i = 1, 2, \ldots, k$.

Proof. The complement $\overline{H}$ is the disjoint union $G_1 \cup G_2 \cup \ldots \cup G_k$. Observe that $L(H) + L(\overline{H}) = nI - J$. Therefore, by the first two parts of Lemma 2.2 we see that if x is an eigenvector of $L(H)$ corresponding to n, then x is also a zero eigenvector of $L(H)$. The conclusion now follows from the third part of Lemma 2.2. □

A graph G is called *split* if its vertex set can be partitioned into two sets, S and C, so that S induces an independent set S and C incudes a clique. If all edges between S and C are present in G, then G will be called the *complete split* graph $CS_{c,s}$, where $c = |C|, s = |S|$. Notice that $CS_{c,s} = \overline{K_s} \vee K_c$.

Let us now list a few facts about signless Laplacians that will prove useful in the sequel.

THEOREM 2.5. *[7] If* $\delta(G) > 0$*, then* $\mu(G) < \delta(G)$.

THEOREM 2.6. *[22] If* G_0 *is a spanning subgraph of* G*, then*

$$\mu(G_0) \leq \mu(G).$$

THEOREM 2.7 (cf. [22]). *If* G *is a graph on* n *vertices, then*

$$\mu(G) = \min_{x \in \mathbb{R}^n - \{0\}} \frac{x^T Q x}{x^T x} = \min_{x \in \mathbb{R}^n - \{0\}} \frac{\sum_{ij \in E(G)} (x_i + x_j)^2}{x^T x}$$

COROLLARY 2.8. *Let* H *be a graph with independent set* $S \subseteq V(G)$*. Let* $T = V(G) - S$ *and supppose that the subgraph induced on* T *has* e *edges. Then:*

$$\mu(H) \leq \frac{4e}{|S| + |T|}.$$

Proof. Define the vector $x \in \mathbb{R}^{t+s}$ by:

$$x_i = \begin{cases} -1 & \text{, if } i \in S \\ 1 & \text{, if } i \in T \end{cases}.$$

Now apply Theorem 2.7 with the x we just defined:

$$\mu(H) \leq \frac{\sum_{ij \in E(G)} (x_i + x_j)^2}{x^T x} = \frac{4e}{|S| + |T|}.$$

□

3. Matrix-theoretic tools. We shall write the eigenvalues of a $n \times n$ Hermitian matrix M in non-decreasing order, *i.e.* $\lambda_1(M) \leq \lambda_2(M) \ldots \leq \lambda_n(M)$. $J_{s,t}, J_t$ or sometimes simply J, will denote the all-ones matrix of a suitable size. The ith standard basis (column) vector will be denoted by e_i. The ith row sum of a matrix A will be denoted by $r_i(A)$.

DEFINITION 3.1. *Let M be a real symmetric matrix whose smallest eigenvalue $\lambda_1(M)$ is simple. If $\lambda_1(M)$ has a positive eigenvector, then M will be called* minpositive.

The class of minpositive matrices is quite wide and includes, for instance, irreducible Z-matrices (or more generally, negatives of eventually positive matrices), irreducible inverse-positive (a.k.a. monotone) matrices, and negatives of certain copositive matrices. Note also that since M is Hermitian, minpositivity of M is equivalent to M^{-1} possessing the *strong Perron-Frobenius property* in the sense of [18].

The next theorem is the weak version of Weyl's inequalities, together with the condition for equality that has been given by Wasin So in [23]. So's condition is valid for the strong version of Weyl's inequalities ([12, Theorem 4.3.7]) as well but the weak version will do for us here.

THEOREM 3.2. *[12, Theorem 4.3.1] Let A, B be Hermitian $n \times n$ matrices. Then for any $k = 1, 2, \ldots, n$ we have:*

$$\lambda_k(A) + \lambda_1(B) \leq \lambda_k(A + B) \leq \lambda_k(A) + \lambda_n(B)$$

For each of these inequalities, equality is satisfied if and only if x is an eigenvector of all three matrices $A, B, A + B$, with the appropriate eigenvalues.

Let us establish a simple result relating the usual and signless Laplacians. It is slightly reminiscent of [13, Theorem 2.1].

THEOREM 3.3. *Let G be a graph with smallest signless Laplacian eigenvalue μ and largest Laplacian eigenvalue λ. Let the minimum degree of G be $\delta = \delta(G)$. Then:*

$$\mu \geq 2\delta - \lambda.$$

Equality obtains if and only if there is a vector x so that:

- $Qx = \mu x$
- $Lx = \lambda x$
- $Dx = \delta x$.

Proof. Clearly $Q + L = 2D$. By applying Theorem 3.2 for $k = 1$ it follows that:

$$\lambda_1(Q + L) \leq \lambda_1(Q) + \lambda_n(L)$$

But $\lambda_1(Q+L) = \lambda_1(2D) = 2\delta$ and therefore $\mu+\lambda \geq 2\delta$. The equality characterization follows from the last part of Theorem 3.2. □

Recall further that the *Schur complement* of the partitioned matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{3.1}$$

is $M/D = A - BD^{-1}C$, assuming that D is invertible.

THEOREM 3.4. *[11] Let M be a Hermitian matrix and let D be a nonsingular principal submatrix of M. Then M is positive semidefinite of and only if D and M/D are both positive semidefinite.*

4. Analyzing the S-Roth property. Let H be a connected non-bipartite graph with a maximal independent set $S \subseteq V(G)$ and let $T = V(G) - S$. Two subgraphs of H are of special interest to us: $G_{H,T}$, the subgraph induced by T and $B_{H,S}$, the bipartite subgraph obtained by deleting all the edges between vertices in T from H. When in no danger of confusion, we will simply write G_H, B_H or even G or B.

Whether H will turn out to be S-Roth will depend on the analysis of $G_{H,T}$ and $B_{H,S}$ and their interplay. Let us now put the discussion into matrix-theoretic terms. Ordering the vertices of H with those in T listed first and those in S listed last, we can write the signless Laplacian $Q(H)$ as:

$$Q(H) = \begin{bmatrix} Q(G) + D_1 & K \\ K^T & D_2 \end{bmatrix},$$

where $Q(G)$ is the signless Laplacian matrix of G and D_1, D_2 are digonal matrices. We shall henceforth write Q instead of $Q(G)$ when in no danger of confusion.

The diagonal entries of D_2 are simply the degrees of the vertices in S. A diagonal entry of D_1 records the number of of vertices in S that are adjacent to the corresponding vertex of T.

Now let $\mu = \mu(H)$ be the smallest eigenvalue of $Q(H)$. Since H is not bipartite, we have $\mu > 0$.

Lemma 4.1. *We have $\mu < \min_{1 \le i \le s} (D_2)_{ii}$ and therefore the matrix $\mu I - D_2$ is invertible.*

Proof. As noted above, the diagonal entries of D_2 are the degrees of the vertices in S. The conclusion follows immediately from Theorem 2.5. □

Let x be an eigenvector corresponding to μ and let us write down the eigenequation, with x partitioned into vectors w, z conformally with the partition of $Q(H)$:

$$\begin{bmatrix} Q + D_1 & K \\ K^T & D_2 \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} = \mu \begin{bmatrix} w \\ z \end{bmatrix}. \tag{4.1}$$

We remark that $w = x(S), z = x(T)$. Multiplying out we have the following system of equations:

$$(Q + D_1)w + Kz = \mu w \tag{4.2}$$

$$K^T w + D_2 z = \mu z \tag{4.3}$$

Lemma 4.1 ensures that equation (4.3) can be solved as:

$$z = (\mu I - D_2)^{-1} K^T w \tag{4.4}$$

We can now give a useful characterization of S-Rothness.

Proposition 4.2. *The graph H is S-Roth if and only if for every eigenvector x corresponding to $\mu(H)$ it holds that $x(S) > 0$ or $x(S) < 0$.*

Proof. One direction is trivial from the definition of an S-Roth graph. For the other direction, consider a partition of x as in (4.1), with $w = x(S)$ and $z = x(T)$. Since H is connected, every vertex in S must have at least neighbour in T. This means that K^T has no zero row. Furthermore, the matrix $\mu I - D_2$ is diagonal and all its diagonal entries are negative, by Lemma 4.1. Therefore, if $w > 0$ it follows from (4.4) that $z < 0$ and we are done. □

Let us now substitute the expression for z found in (4.4) into (4.2):

$$(Q + D_1)w + K(\mu I - D_2)^{-1}K^T w = \mu w$$

In other words, (μ, w) is an eigenpair of the following matrix:

$$Q_\mu = Q + D_1 + K(\mu I - D_2)^{-1}K^T. \tag{4.5}$$

LEMMA 4.3. *μ is the smallest eigenvalue of Q_μ.*
Proof. Consider the shifted matrix $Q(H) - \mu I$:

$$Q(H) - \mu I = \begin{bmatrix} Q + D_1 - \mu I & K \\ K^T & D_2 - \mu I \end{bmatrix},$$

Lemma 4.1 enables us to take the Schur complement of $Q(H) - \mu I$ by its bottom-right block:

$$(Q(H) - \mu I)/(D_2 - \mu I) = Q + D_1 - \mu I + K(\mu I - D_2)^{-1}K^T = Q_\mu - \mu I$$

The matrix $Q(H) - \mu I$ is clearly positive semidefinite and thus by Theorem 3.4 the matrix $Q_\mu - \mu I$ is positive semidefinite as well. This means that all eigenvalues of Q_μ are greater than or equal to μ. □

We have thus arrived at an alternative characterization of the S-Roth property.

THEOREM 4.4. *H is S-Roth if and only if Q_μ is a minpositive matrix.*

Proof. Immediate from Lemma 4.3 and Proposition 4.2. □

5. A combinatorial sufficient condition for S-Rothness. We would now like to formulate simple combinatorial conditions on $B_{H,S}$ and $G_{H,T}$ that will ensure that the graph H is S-Roth, drawing upon Theorem 4.4. We continue to assume in this section that the vertices of H are sorted so that those of T come first, then those of S.

PROPOSITION 5.1. *Let i, j be distinct indices in $\{1, 2, \dots, t\}$ and let $N_{i,j} = \{k \in S | k \sim_B i, k \sim_B j\}$ be the set of their common neighbours in S. Then:*

$$(K(D_2 - \mu I)^{-1}K^T)_{ij} = \sum_{k \in N_{ij}} \frac{1}{d_B(k) - \mu}.$$

Proof. Immediate, upon observing that the diagonal entries of D_2 are the degrees of the vertices in S. □

We can now state the main result of this section. Recall that a real matrix is called a Z-matrix if all of its offdiagonal entries are nonpositive.

THEOREM 5.2. *Let H be a connected non-bipartite graph with a maximal independent set $S \subseteq V(H)$. Suppose that*

- *For all $ij \in E(G)$:*

$$\sum_{k \in N_{ij}} \frac{1}{d_B(k)} \geq 1$$

and that

- *For all $ij \notin E(G)$:*

$$N_{ij} \neq \emptyset.$$

Then H is S-Roth.

Proof. Consider the ijth off-diagonal entry of Q_μ: by Equation (4.5) and Proposition 5.1 it is equal to $1-\sum_{k\in N_{ij}}\frac{1}{d_B(k)-\mu}$ if $ij\in E(G)$ and to $-\sum_{k\in N_{ij}}\frac{1}{d_B(k)-\mu}$ if $ij\notin E(G)$. Therefore, by our assumptions (and the fact that $\mu>0$), it is negative in both cases. Thus Q_μ is a Z-matrix all of whose off-diagonal entries are strictly negative, ergo it is minpositive. We are done by Theorem 4.4. □

COROLLARY 5.3. *Let $c^B=\max_{k\in S} d_B(k)$ and suppose that $|N_{ij}|\geq c^B$ for all $ij\in E(G)$ and that $N_{ij}\neq\emptyset$ for all distinct $ij\notin E(G)$. Then H is S-Roth.*

Proof. The condition $|N_{ij}|\geq c^B$ clearly implies that $\sum_{k\in N_{ij}}\frac{1}{d_B(k)}\geq |N_{ij}|\frac{1}{c^B}\geq 1$. □

COROLLARY 5.4. *If $d_B(i)\geq\frac{t+s}{2}$ for every vertex i in T, then H is S-Roth.*

Proof. A simple counting argument shows that in this case $|N_{ij}|\geq t$ for all i,j. On the other hand, clearly $c^B\leq t$. □

COROLLARY 5.5. *Supppose that $B_{H,S}=K_{s,t}$. If $s\geq t$, then H is S-Roth.*

Proof. In this case $|N_{ij}|=s$ for all i,j and $c^B=t$. □

REMARK 5.6. *The difference between Theorem 5.2 and Corollary 5.3 is that the theorem posits a more refined "local" condition, whereas the corollary operates via a cruder "global" condition.*

Our focus in this section was mainly on $B_{H,S}$. We now fix $B_{H,S}=K_{s,t}$, as in Corollary 5.5 and turn to study in greater detail the effect of $G_{H,T}$ on the S-Rothness of G. If $s\geq t$ we already know that H is S-Roth. Some results about the case $s<t$ will be presented in the next section.

Note that when $B=K_{s,t}$ it does not matter in which way G is "glued" to B, and S-Rothness depends only on G in itself. This is not true for other fixed graphs B of course.

6. A case study. In order to better understand Theorem 5.2 and its scope of applicability we examined three special cases $s\in\{5,7,9\}, t=4$ with $G=K_4$ for all possible *connected* bipartite graphs B. This has been made possible by using data made publicly available by Gordon Royle [21].

Each row of Table 6.1 records information about one value of s. The seven columns enumerate the following: the value of s, the total number of connected bipartite graphs, the number of S-Roth graphs amongst them, the number of graphs that satisfy the conditions of Theorem 5.2, the number graphs such that their Q_μ is minpositive, and the number of graphs such that Q_μ^{-1} is entrywise positive.

TABLE 6.1
S-Rothness for small bipartite graphs

s	# graphs	S-Roth	Theorem 5.2	Q_μ is an M-matrix	$Q_\mu^{-1}>0$
5	558	64	4	23	35
7	5375	823	85	283	515
9	36677	8403	1234	3155	6054

Some observations from Table 6.1: Theorem 5.2 becomes more powerful as s increases with respect to t. On the other hand, we see that there are many cases when Q_μ is an M-matrix which are not accounted for by Theorem 5.2; a possible way to extend its coverage would be by incorporating into the argument some lower bounds on $\mu(H)$. These are, however, notoriously hard to come by.

Note that our concepts make sense for disconnected B as well but we have chosen to omit them from this study, hopefully with little loss.

Note also that by choosing G to be a clique we are, so to say, taxing to the utmost Theorem 5.2.

Let us now consider some examples to illustrate the possibilities. All examples will be drawn from the case $s = 7, t = 4, G = K_4$.

EXAMPLE 6.1. *Let B be graph # 4530 in Royle's catalogue. In this case the matrix K is given by:*

$$K = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The degrees $d_B(1), d_B(2), \ldots, d_B(7)$ are $4, 4, 4, 4, 1, 1, 1$ (read off as the column sums of K). The condition of Theorem 5.2 is met as $N_{ij} = \{1, 2, 3, 4\}$ for all i, j and therefore $\sum_{k \in N_{ij}} \frac{1}{d_B(k)} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$.

And indeed, the matrix Q_μ can be computed as:

$$Q_\mu = \begin{pmatrix} 5.8123 & -0.18774 & -0.18774 & -0.18774 \\ -0.18774 & 5.8123 & -0.18774 & -0.18774 \\ -0.18774 & -0.18774 & 5.8123 & -0.18774 \\ -0.18774 & -0.18774 & -0.18774 & 0.65427 \end{pmatrix}, \mu = 0.63226.$$

The eigenvector x is:

$$x = \begin{pmatrix} 0.008 \\ 0.008 \\ 0.008 \\ 0.2057 \\ -0.0682 \\ -0.0682 \\ -0.0682 \\ -0.0682 \\ -0.5594 \\ -0.5594 \\ -0.5594 \end{pmatrix}.$$

EXAMPLE 6.2. *Now consider an example where Q_μ is an M-matrix but the condition of Theorem 5.2 is not met. Let G be graph # 5104 in Royle's catalogue. We have*

$$K = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 \end{pmatrix},$$

and

$$[d_B(1), d_B(2), \ldots, d_B(7)] = [4, 4, 3, 3, 2, 3, 1].$$

Taking $i = 1, j = 2$ *we observe that* $N_{12} = \{1, 2, 3\}$ *and therefore* $\sum_{k \in N_{12}} \frac{1}{d_B(k)} = \frac{1}{4} + \frac{1}{4} + \frac{1}{3} = \frac{5}{6} < 1$. *However,*

$$Q_\mu = \begin{pmatrix} 5.6058 & -0.08776 & -0.93542 & -0.54653 \\ -0.08776 & 0.88934 & -0.08776 & -0.54653 \\ -0.93542 & -0.08776 & 5.6058 & -0.54653 \\ -0.54653 & -0.54653 & -0.54653 & 5.9947 \end{pmatrix}$$

is an M*-matrix* ($\mu = 0.82028$).

EXAMPLE 6.3. *Consider now* B *which is graph # 3503 in Royle's catalogue.*

$$K = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$[d_B(1), d_B(2), \ldots, d_B(7)] = [4, 2, 2, 2, 2, 2, 2].$$

Now $N_{12} = N_{13} = N_{23} = \frac{1}{4}$ *and* $N_{14} = N_{24} = N_{34} = 1\frac{1}{4}$. *Although* Q_μ *has many positive entries, its inverse is nevertheless positive and therefore* Q_μ *is minpositive. In this case* $\mu = 1.0922$. *In fact,*

$$Q_\mu = \begin{pmatrix} 3.453 & 0.6561 & 0.6561 & -1.547 \\ 0.6561 & 3.453 & 0.6561 & -1.547 \\ 0.6561 & 0.6561 & 3.453 & -1.547 \\ -1.547 & -1.547 & -1.547 & 3.0468 \end{pmatrix},$$

$$Q_\mu^{-1} = \begin{pmatrix} 0.37674 & 0.019201 & 0.019201 & 0.21078 \\ 0.019201 & 0.37674 & 0.019201 & 0.21078 \\ 0.019201 & 0.019201 & 0.37674 & 0.21078 \\ 0.21078 & 0.21078 & 0.21078 & 0.64927 \end{pmatrix},$$

EXAMPLE 6.4. B *is now graph # 1447 in Royle's catalogue. Then*

$$K = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

and

$$[d_B(1), d_B(2), \ldots, d_B(7)] = [3, 2, 3, 2, 2, 1, 1].$$

Since $N_{12} = 0$ *we cannot apply Theorem 5.2. Also,* Q_μ *is not monotone. Nevertheless, it can be found by computation that* $w > 0$. *Indeed, we have in this case:*

$$Q_\mu = \begin{pmatrix} 3.8172 & 1 & 0.57038 & -0.18282 \\ 1 & 3.8172 & 0.57038 & -0.18282 \\ 0.57038 & 0.57038 & 4.3876 & -0.61244 \\ -0.18282 & -0.18282 & -0.61244 & 0.77727 \end{pmatrix},$$

$$w = \begin{pmatrix} 0.0047565 \\ 0.0047565 \\ 0.033593 \\ 0.21264 \end{pmatrix}, \mu = 0.67234.$$

We remark that $Q_\mu^{-6} > 0$ in this case but there seems to be no easy combinatorial interpretation of this fact.

7. The case $B_{H,S} = K_{s,t}$ for $s < t$, Part I. First note that if $B_{H,S} = K_{s,t}$, then $H = \overline{K_s} \vee G$. From this point we shall consider only graphs of this form. When we say that H is S-Roth we shall mean that S is the set of vertices inducing the $\overline{K_s}$.

Since $B = K_{s,t}$, we have now $K = J_{t,s}, D_1 = sI_t, D_2 = tI_s$ and therefore the definition of the matrix Q_μ in Equation (4.5) simplifies to:

$$Q_\mu = Q + sI - \frac{s}{t-\mu}J. \tag{7.1}$$

DEFINITION 7.1. *Let $H = \overline{K_s} \vee G$. Let $\mu = \mu(H)$ be the smallest signless Laplacian eigenvalue of H. Define the quantity $\alpha_H(G)$ as:*

$$\alpha_H(G) = \frac{s}{t-\mu}.$$

LEMMA 7.2. *If $t > s$ and $\alpha_H(G) > 1$, then H is S-Roth.*

Proof. If $\alpha_H(G) > 1$, then by (7.1) Q_μ is a Z-matrix all of whose offdiagonal entries are strictly negative. Therefore it is minpositive and the conclusion follows immediately from Theorem 4.4. □

REMARK 7.3. *Notice that $\alpha_H(G) = 1$ is equivalent to $\mu(H) = t-s$ and $\alpha_H(G) > 1$ is equivalent to $\mu(H) > t - s$.*

THEOREM 7.4. *Let $H = \overline{K_s} \vee G$, with $t > s$. Suppose that one of the following cases holds:*

- *(A) $\delta(G) > t - s$; OR*
- *(B) $\delta(G) = t - s$ and $\overline{G}$ is connected.*

Then H is S-Roth.

Proof. The degrees of the vertices in S all equal t. On the other hand, the degrees of the vertices in T are all at least $s + \delta(G)$. Since we assumed $\delta(G) \geq t - s$ in both cases, we have that $\delta(H) \geq t$ in both of them, with equality obtaining in case (B).

Now by Lemma 2.3 and Theorem 3.3 we have that

$$\mu \geq 2t - (t + s) = t - s.$$

Therefore, $\alpha_H(G) \geq 1$ and we see that Q_μ is a Z-matrix.

In case (A) we have $\alpha > 1$ and so we are done by Lemma 7.2. Otherwise, $\alpha_H(G) = 1$ and $Q_\mu = Q + sI - J$ is a Z-matrix whose offdiagonal zero pattern is exactly the same as that of the adjacency matrix of $\overline{G}$. Therefore, if $\overline{G}$ is connected, then Q_μ is irreducible and thus a minpositive matrix. □

If $\delta(G) = t - s$ and $\overline{G}$ is disconnected, then H may fail to be S-Roth, as Example 7.5 will illustrate.

EXAMPLE 7.5. *Let $t = 6, s = 4, G = K_{4,2}$. In this case, we have $\delta(G) = t-s = 2$, but the smallest eigenvector of $Q(H)$ has two zero entries.*

Indeed, in this case we have:

$$Q_\mu = \begin{pmatrix} 5 & -1 & -1 & -1 & 0 & 0 \\ -1 & 5 & -1 & -1 & 0 & 0 \\ -1 & -1 & 5 & -1 & 0 & 0 \\ -1 & -1 & -1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & -1 \\ 0 & 0 & 0 & 0 & -1 & 7 \end{pmatrix}$$

and $\mu = t - s = 2, w = [1,1,1,1,0,0]^T$.

Nevertheless, it is possible to completely characterize S-Rothness when $\delta(G) = t - s$ and $\overline{G}$ is disconnected.

THEOREM 7.6. *Let* $H = \overline{K_s} \vee G$ *with* $t > s$ *and suppose that* $\delta(G) = t - s$*, that* $\overline{G}$ *is disconnected, and that* $G = G_1 \vee G_2 \vee \ldots \vee G_k$ *is a maximal join decomposition. Then* H *is* S*-Roth if and only if for each* $j = 1, 2, \ldots, k$ *there is a vertex* $v_j \in V(G_j)$ *such that* $d_G(v_j) > t - s$.

Proof. Suppose first without loss of generality that for every vertex $v \in V(G_1)$ it holds that $d_G(v) = t - s$. Define a vector $x \in \mathbb{R}^{|V(H)|}$ by:

$$x_i = \begin{cases} -|V(G_1)| & \text{, if } i \in S \\ s & \text{, if } i \in V(G_1) \\ 0 & \text{, otherwise} \end{cases}.$$

It is easy to verify then that $Q(H)x = (t - s)x$. Therefore, x is an eigenvector of $Q(H)$ corresponding to the eigenvalue $t - s$. From the proof of Theorem 7.4 we know that $\mu \geq t - s$ and therefore $\mu = t - s$. But since x has zero entries, we deduce that H is not S-Roth.

Suppose now that there is a vertex $v_j \in V(G_j)$ such that $d_G(v_j) > t - s$ for every j. We can also assume that $\mu = t - s$ since otherwise we would be done by Lemma 7.2. Let x be an eigenvector of $Q(H)$ corresponding to μ.

From Theorem 3.3 we see that $(x, 2t)$ is an eigenpair of $D(H)$ and that $(x, t+s)$ is an eigenpair of $L(H)$. Now, the first statement immediately implies that x is nonzero only on vertices of degree t in H, whereas the second statement implies, via Lemma 2.2 that x is contant over S and over $V(G_i)$ for every i. Put together with our assumption, these two observations imply that $x = 0$. This is a contradiction, and therefore the case $m = t - s$ is impossible. Hence, $\mu > t - s$ and H is S-Roth by Lemma 7.2. □

THEOREM 7.7. *Let* $H = \overline{K_s} \vee G$*, with* $2 \leq s < t$*. If* G *has at least* $t - s + 1$ *vertices of degree* $t - 1$*, then* H *is* S*-Roth.*

Proof. It is easy to see that the assumption on G is equivalent to the existence in G of a spanning subgraph isomorphic to $CS_{t-s+1,s-1}$. Let $H_0 = \overline{K_s} \vee CS_{t-s+1,s-1}$. We claim that it is enough to prove that $\alpha_{H_0} > 1$. Indeed, since H_0 is a spanning subgraph of H, it follows that $\alpha_H \geq \alpha_{H_0} > 1$ and we are done by Lemma 7.2.

Consider then the signless Laplacian Q_0 of the graph H_0. Ordering the vertices as: the clique of G, then the independent set of G, then S, we see that Q_0 has the following form:

$$Q_0 = \begin{bmatrix} (t+s-2)I + J & J & J \\ J & (t+1)I & J \\ J & J & tI \end{bmatrix}$$

If we can show that $Q_0-(t-s)I$ is positive definite, then we shall know that $\mu(H_0) > t-s$ (which is equivalent to $\alpha_{H_0} > 1$). To that end, take the Schur complement of $Q_0-(t-s)I$ by its bottom-right block:

$$Q_0-(t-s)I = \begin{bmatrix} (2s-2)I+J & J & J \\ J & (s+1)I & J \\ J & J & sI \end{bmatrix},$$

$$(Q_0-(t-s)I)/(sI) = \begin{bmatrix} (2s-2)I & 0 \\ 0 & (s+1)I-J \end{bmatrix}.$$

That $Q_0-(t-s)I$ is positive definite is now obvious from Theorem 3.4. □

We can also deduce a rudimentary extremal-type result:

Corollary 7.8. *$H=\overline{K_s}\vee G$, with $2\le s<t$. If G has at least $\binom{t}{2}-\lfloor\frac{s-1}{2}\rfloor$ edges, then H is S-Roth.*

Proof. Consider G as K_t from which a number of edges have been deleted. The deletion of each edge lowers the degrees of at most two vertices below $t-1$ so that the resulting graph satisfies the conditions of Theorem 7.7. □

8. Interlude - A new analysis of S-Rothness. We shall now revisit the analysis of the eigenequations (4.2) and (4.3), with the vectors w and z trading roles this time. We will only consider the case $B=K_{s,t}$ here so we may at once take $K=J_{t,s}$ and $D_1=sI, D_2=tI$.

Now we introduce a new matrix:

$$R_\mu = (Q+sI-\mu I).$$

We will make a crucial assumption here: that $s>\mu$. In this case R_μ is positive definite.

Consider first Equation (4.2).If we assume that the matrix $(Q+sI-\mu I)$ is non-singular, we can solve it as:

$$w=-R_\mu^{-1}Jz. \tag{8.1}$$

Substituting (8.1) into (4.3) we obtain:

$$-J^T R_\mu^{-1}Jz+tz=\mu z.$$

Now $J^T R_\mu^{-1}J$ is clearly a multiple of J, namely γJ where γ is the sum of all entries in R_μ^{-1}. Note that $\gamma>0$, since R_μ^{-1} is also positive definite. Therefore we can write:

$$\gamma Jz=(t-\mu)z. \tag{8.2}$$

Observe now that $Jz=\sigma_z\mathbf{1}$ where σ_z is the sum of all entries in z. It is impossible to have $\sigma_z=0$ since it would imply $z=0$ by (8.2) and consequently also $w=0$ by (8.1) - a contradiction. Therefore we infer that

$$z=\frac{\gamma\sigma_z}{t-\mu}\mathbf{1}. \tag{8.3}$$

Now assume without loss of generality that $z<0$; we consult again Equation (8.1) and see that w is a positive miltiple of the vector of row sums of R_μ. We can summarize our findings as follows:

THEOREM 8.1. *Let $H = \overline{K_s} \vee G$. Suppose that R_μ is positive definite. Then H is S-Roth if and only if the row sums of R_μ^{-1} are all positive.*

REMARK 8.2. *If $\mu(H) = s$ and G is bipartite, then R_μ is singular and we can not use it to analyze S-Rothness. For example, this happens when $s = 3, t = 14, G = C_{14}$. In that example we have $z = 0$.*

9. Some more matrix tools. The preceding section saw the injection of inverse matrices into the discussion. Their analysis will require two more tools which we present here. The matrices we shall deal with in the next section will be of the form $Q(C_k) + \lambda I$ and are easily seen to be strictly diagonally dominant.

The study of strictly diagonally dominant matrices is of course an old and venerable enterprise. The fact, crucially useful to us here, that some measure of the diagonal dominance carries over to the inverse seems to have been noticed first by Ostrowski [19] in 1952. We will use a slightly more recent result that quantifies this statement:

THEOREM 9.1. *[14, Theorem 2.4] Let A be a strictly diagonally dominant matrix. Let $\widetilde{A} = A^{-1} = (\widetilde{a}_{ij})$. Then we have:*

$$|\widetilde{a}_{ji}| \le \max_{l \ne i} \left\{ \frac{|a_{li}|}{|a_{ll}| - \sum_{k \ne l,i} |a_{lk}|} \right\} |\widetilde{a}_{ii}|, \quad \textit{for all} \quad j \ne i.$$

We shall also use a remarkable result of Bai and Golub, that gives another kind of insight into A^{-1}.

THEOREM 9.2. *[1] Let $A \in \mathbb{R}^{n,n}$ symmetric positive definite matrix whose eigenvalues lie in $[a, b]$, with $a > 0$. Furthermore, let $m_1 = \operatorname{Tr} A, m_2 = ||A||_F^2$. Then:*

$$\begin{bmatrix} m_1 & n \end{bmatrix} \begin{bmatrix} m_2 & m_1 \\ b^2 & b \end{bmatrix}^{-1} \begin{bmatrix} n \\ 1 \end{bmatrix} \le \operatorname{Tr} A^{-1}$$

and

$$\operatorname{Tr} A^{-1} \le \begin{bmatrix} m_1 & n \end{bmatrix} \begin{bmatrix} m_2 & m_1 \\ a^2 & a \end{bmatrix}^{-1} \begin{bmatrix} n \\ 1 \end{bmatrix}.$$

Now let $A = Q(C_k) + \lambda I$, with $\lambda > 0$. We immediately observe that A is both symmetric positive definite and strictly diagonally dominant. Notice also that A is a circulant matrix and therefore A^{-1} is also circulant (cf. [8, p. 74]) and therefore all diagonal entries of A^{-1} are equal to $\operatorname{Tr} A^{-1}/k$. Combining Theorem 9.1 with the upper bound of Theorem 9.2, we have the following bounds on the entries of A:

LEMMA 9.3. *Let $A = Q(C_k) + \lambda I, \lambda > 0$ and let $\widetilde{A} = A^{-1}$. Then:*

$$\widetilde{a}_{ii} \le \frac{\lambda + 1}{\lambda(\lambda + 3)}$$

and

$$|\widetilde{a}_{ij}| \le \frac{1}{\lambda + 1} |\widetilde{a}_{ii}|, \forall j \ne i.$$

Proof. The first inequality follows from Theorem 9.2, upon taking $a = \lambda, m_1 = n(2 + \lambda), m_2 = n(2 + \lambda)^2 + 2n$ and doing some algebra. The second inequality follows from Theorem 9.1 quite easily, upon careful index-chasing. ∎

10. The case $B_{H,S} = K_{s,t}$ for $s < t$, Part II. In Section 7 we studied the situation when G is relatively dense. Now we turn to investigate the case when G is sparse. It will be seen that (at least assuming $B_{H,S} = K_{s,t}$) the matrix R_μ will provide a handier tool in this case than Q_μ. The reason is that Theorem 8.1 uses a simpler property than Theorem 4.4 - provided that the inverse R_μ^{-1} can be described well; and indeed, if G is sparse, then the structure of R_μ^{-1} can often be inferred from that of R_μ.

The main result of this section is:

THEOREM 10.1. *Let $t > s \geq 6$ and let $H = \overline{K_s} \vee G$. If $\Delta(G) \leq 2$, then H is S-Roth.*

Proof. Obviously, G is a disjoint union of cycles, paths and isolated vertices. Therefore it has at most t edges. By Corollary 2.8, we see that $\mu(H) \leq \frac{4t}{t+s} < 4$. Thus we may use Theorem 8.1. The matrix R_μ is block-diagonal, each block corresponding to a connected component of G. We can write

$$R_\mu = A_1 \oplus \ldots \oplus A_k \oplus B_1 \oplus \ldots \oplus B_m \oplus (s - \mu)I,$$

with the A_is corresponding to the cycles, the B_js to the paths and the last summand lumping together all isolated vertices, if there are any. Clearly:

$$R_\mu^{-1} = A_1^{-1} \oplus \ldots \oplus A_k^{-1} \oplus B_1^{-1} \oplus \ldots \oplus B_m^{-1} \oplus (s - \mu)^{-1}I,$$

and so we need to show that the row sums of each A_i^{-1} and each B_j^{-1} are positive.

The row sums of A_i are all equal to $4 + s - \mu$ and therefore the row sums of A_i^{-1} are all equal to $(4 + s - \mu)^{-1}$ and thus are positive. Denote for further use $\beta = (4 + s - \mu)^{-1}$.

Finally, let us consider a B_j. It corresponds to a path component of G on, say, k vertices. Had this component been a cycle its matrix A would have had positive row sums, by the preceding argument. We shall want to write B_j as a rank-one modification of A and to show that the row sums remain positive:

$$B_j = A - E, \quad E = (e_1 + e_k)(e_1 + e_k)^T.$$

The Sherman-Morrison formula then shows that:

$$B_j^{-1} = A^{-1} + \frac{A^{-1}EA^{-1}}{1 - (e_1 + e_k)^T A^{-1}(e_1 + e_k)}.$$

Now, as $A = Q(C_k) + (s - \mu)I$, we can bring into play the observations made in Section 9. We shall write $\widetilde{A} = A^{-1}$ and denote by d the common value of the diagonal entries of $\widetilde{A}$.

The expression $(e_1 + e_k)^T A^{-1}(e_1 + e_k)$ is equal to the sum of the four corner entries of A^{-1}, that is to say, to $2d + 2\widetilde{a}_{1k}$. Therefore:

$$B_j^{-1} = \widetilde{A} + \frac{\widetilde{A}E\widetilde{A}}{1 - 2(d + \widetilde{a}_{1k})}. \tag{10.1}$$

Taking the row sums of both sides of (10.1) and bearing in mind that $\widetilde{A}\mathbf{1} = \beta\mathbf{1}$, we obtain that the vector of row sums of B_j^{-1} is equal to $2\beta(\widetilde{A}e_1 + \widetilde{A}e_t)$. Therefore:

$$r_i(B_j^{-1}) = \beta\left(1 + \frac{2(d + \widetilde{a}_{1k})}{1 - 2(d + \widetilde{a}_{1k})}\right), i \in \{1, t\} \tag{10.2}$$

and

$$r_i(B_j^{-1}) = \beta\left(1 + \frac{2(b_{i1} + \widetilde{a}_{ik})}{1 - 2(d + \widetilde{a}_{1k})}\right), 2 \leq 2 \leq t - 1. \tag{10.3}$$

Finally, we use Lemma 9.3 to estimate $d, \widetilde{a}_{1k}, \widetilde{a}_{i1}, \widetilde{a}_{ik}$: since $\lambda = s - \mu > 2$, we have that $d < 0.3$ and $|\widetilde{a}_{1k}|, |\widetilde{a}_{i1}|, |\widetilde{a}_{ik}| < 0.1$. Furthermore, a simple determinantal calculation shows that $\widetilde{a}_{1t} < 0$. This implies that

$$\frac{2(d + \widetilde{a}_{1k})}{1 - 2(d + \widetilde{a}_{1k})} > 0, \left|\frac{2(\widetilde{a}_{i1} + \widetilde{a}_{ik})}{1 - 2(d + \widetilde{a}_{1k})}\right| < \frac{0.4}{0.4} = 1$$

and therefore $r_i(B_j^{-1})$ is always positive. □

REMARK 10.2. *The conclusion of Theorem 10.1 remains true for $s = 5$ as well but the proof we gave will not go through. The difficulty is posed by the components of G that are paths and it can be handled by a different argument that uses the fact that B_j is then tridiagonal and that therefore B_j^{-1} is a Green's matrix (cf. [16]); we omit here the details, which are somewhat tedious.*

On the other hand, S-Rothness may fail altogether when $s = 4$. Indeed, it can be easily verified that, say, $\overline{K_4} \vee P_{60}$ is not S-Roth.

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