Discrete-time population growth models
Discrete-time systems

In continuous-time models $t \in \mathbb{R}$. Another way to model natural phenomena is to consider equations of the form

$$x_{t+1} = f(x_t),$$

where $t \in \mathbb{N}$ or $\mathbb{Z}$, that is, $t$ takes values in a discrete valued (countable) set

Time could for example be days, years, etc.
Suppose we have a system in the form

\[ x_{t+1} = f(x_t), \]

with initial condition given for \( t = 0 \) by \( x_0 \). Then,

\[ x_1 = f(x_0) \]
\[ x_2 = f(x_1) = f(f(x_0)) \triangleq f^2(x_0) \]
\[ \vdots \]
\[ x_k = f^k(x_0). \]

The \( f^k = f \circ f \circ \cdots \circ f \) are called the **iterates** of \( f \).
Fixed points

Definition (Fixed point)
Let $f$ be a function. A point $p$ such that $f(p) = p$ is called a **fixed point** of $f$.

Indeed, if $f(p) = p$, then

\[
\begin{align*}
f(p) &= p \\
f(f(p)) &= f(p) = p \\
f(f(f(p))) &= f(p) = p \\
&\vdots \\
f^k(p) &= p \quad \forall k \in \mathbb{N}
\end{align*}
\]

so the system is *fixed* (stuck) at $p$. ..
Theorem
Consider the closed interval \( I = [a, b] \). If \( f : I \rightarrow I \) is continuous, then \( f \) has a fixed point in \( I \).

Theorem
Let \( I \) be a closed interval and \( f : I \rightarrow \mathbb{R} \) be a continuous function. If \( f(I) \supset I \), then \( f \) has a fixed point in \( I \).
Periodic points

Definition (Periodic point)
Let \( f \) be a function. If there exists a point \( p \) and an integer \( n \) such that
\[
f^n(p) = p, \quad \text{but} \quad f^k(p) \neq p \quad \text{for} \quad k < n,
\]
then \( p \) is a periodic point of \( f \) with (least) period \( n \) (or a \( n \)-periodic point of \( f \)).

Thus, \( p \) is a \( n \)-periodic point of \( f \) iff \( p \) is a 1-periodic point of \( f^n \).
Stability of fixed points, of periodic points

Theorem

Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuously differentiable function (that is, differentiable with continuous derivative, or \( C^1 \)), and \( p \) be a fixed point of \( f \).

1. If \( |f'(p)| < 1 \), then there is an open interval \( I \ni p \) such that \( \lim_{k \to \infty} f^k(x) = p \) for all \( x \in I \).

2. If \( |f'(p)| > 1 \), then there is an open interval \( I \ni p \) such that if \( x \in I, x \neq p \), then there exists \( k \) such that \( f^k(x) \notin I \).

Definition

Suppose that \( p \) is a \( n \)-periodic point of \( f \), with \( f : \mathbb{R} \to \mathbb{R} \in C^1 \).

- If \( |(f^n)'(p)| < 1 \), then \( p \) is an attracting periodic point of \( f \).
- If \( |(f^n)'(p)| > 1 \), then \( p \) is an repelling periodic point of \( f \).
Robert May

- Born 1938 in Sydney, Australia
- 1962, Professor of Theoretical Physics, University of Sydney
- 1973, Professor of Zoology, Princeton University
- 1988, Professor, Imperial College and University of Oxford

Known for

*Simple mathematical models with very complicated dynamics*  
(Nature, 1976)
The logistic map

The logistic map is, for \( t \geq 0 \),

\[
N_{t+1} = rN_t \left( 1 - \frac{N_t}{K} \right). \tag{DT1}
\]

To transform this into an initial value problem, we need to provide an initial condition \( N_0 \geq 0 \) for \( t = 0 \).
Consider the equation (DT1), which for convenience we rewrite as

\[ N_{t+1} = rN_t(1 - N_t), \]  

(DT2)

where \( r \) is a parameter in \( \mathbb{R}_+ \), and \( N \) will typically be taken in \([0, 1] \). Let

\[ f_r(x) = rx(1 - x). \]

The function \( f_r \) is called a **parametrized family** of functions.
Bifurcations

Definition (Bifurcation)

Let $f_\mu$ be a parametrized family of functions. Then there is a bifurcation at $\mu = \mu_0$ (or $\mu_0$ is a bifurcation point) if there exists $\varepsilon > 0$ such that, if $\mu_0 - \varepsilon < a < \mu_0$ and $\mu_0 < b < \mu_0 + \varepsilon$, then the dynamics of $f_a(x)$ are “different” from the dynamics of $f_b(x)$.

An example of “different” would be that $f_a$ has a fixed point (that is, a 1-periodic point) and $f_b$ has a 2-periodic point.
Consider the simplified version (DT2),

\[ N_{t+1} = rN_t(1 - N_t) \triangleq f_r(N_t). \]

**Are solutions well defined?** Suppose \( N_0 \in [0, 1] \), do we stay in \([0, 1]\)? \( f_r \) is continuous on \([0, 1]\), so it has a extrema on \([0, 1]\). We have

\[ f'_r(x) = r - 2rx = r(1 - 2x), \]

which implies that \( f_r \) increases for \( x < 1/2 \) and decreases for \( x > 1/2 \), reaching a maximum at \( x = 1/2 \).

\( f_r(0) = f_r(1) = 0 \) are the minimum values, and \( f(1/2) = r/4 \) is the maximum. Thus, if we want \( N_{t+1} \in [0, 1] \) for \( N_t \in [0, 1] \), we need to consider \( r \leq 4 \).
Note that if \( N_0 = 0 \), then \( N_t = 0 \) for all \( t \geq 1 \).

Similarly, if \( N_0 = 1 \), then \( N_1 = 0 \), and thus \( N_t = 0 \) for all \( t \geq 1 \).

This is true for all \( t \): if there exists \( t_k \) such that \( N_{t_k} = 1 \), then \( N_t = 0 \) for all \( t \geq t_k \).

This last case might occur if \( r = 4 \), as we have seen.

Also, if \( r = 0 \) then \( N_t = 0 \) for all \( t \).

For these reasons, we generally consider

\[ N \in (0, 1) \]

and

\[ r \in (0, 4). \]
Fixed points: existence

Fixed points of (DT2) satisfy $N = rN(1 - N)$, giving:

- $N = 0$;
- $1 = r(1 - N)$, that is, $p \triangleq \frac{r - 1}{r}$.

Note that $\lim_{r \to 0^+} p = 1 - \lim_{r \to 0^+} 1/r = -\infty$, $\frac{\partial}{\partial r} p = 1/r^2 > 0$ (so $p$ is an increasing function of $r$), $p = 0 \iff r = 1$ and $\lim_{r \to \infty} p = 1$. So we come to this first conclusion:

- 0 always is a fixed point of $f_r$.
- If $0 < r < 1$, then $p$ takes negative values so is not relevant.
- If $1 < r < 4$, then $p$ exists.
Stability of the fixed points

Stability of the fixed points is determined by the (absolute) value $f'_r$ at these fixed points. We have

$$|f'_r(0)| = r,$$

and

$$|f'_r(p)| = \left| r - 2r \frac{r - 1}{r} \right| = |r - 2(r - 1)| = |2 - r|$$

Therefore, we have

- if $0 < r < 1$, then the fixed point $N = p$ does not exist and $N = 0$ is attracting,
- if $1 < r < 3$, then $N = 0$ is repelling, and $N = p$ is attracting,
- if $r > 3$, then $N = 0$ and $N = p$ are repelling.
Bifurcation diagram for the discrete logistic map
Another bifurcation

Thus the points $r = 1$ and $r = 3$ are bifurcation points. To see what happens when $r > 3$, we need to look for period 2 points.

$$f_r^2(x) = f_r(f_r(x))$$
$$= rf_r(x)(1 - f_r(x))$$
$$= r^2 x(1 - x)(1 - rx(1 - x)). \quad (1)$$

0 and $p$ are points of period 2, since a fixed point $x^*$ of $f$ satisfies $f(x^*) = x^*$, and so, $f^2(x^*) = f(f(x^*)) = f(x^*) = x^*$. This helps localizing the other periodic points. Writing the fixed point equation as

$$Q(x) \overset{\Delta}{=} f_r^2(x) - x = 0,$$

we see that, since 0 and $p$ are fixed points of $f_r^2$, they are roots of $Q(x)$. Therefore, $Q$ can be factorized as

$$Q(x) = x(x - p)(-r^3x^2 + Bx + C),$$
Substitute the value \((r - 1)/r\) for \(p\) in \(Q\), develop \(Q\) and (1) and equate coefficients of like powers gives

\[
Q(x) = x \left( x - \frac{r - 1}{r} \right) \left( -r^3 x^2 + r^2 (r + 1)x - r(r + 1) \right). \tag{2}
\]

We already know that \(x = 0\) and \(x = p\) are roots of (2). So we search for roots of

\[
R(x) := -r^3 x^2 + r^2 (r + 1)x - r(r + 1).
\]

Discriminant is

\[
\Delta = r^4 (r + 1)^2 - 4r^4 (r + 1)
\]
\[
= r^4 (r + 1)(r + 1 - 4)
\]
\[
= r^4 (r + 1)(r - 3).
\]

Therefore, \(R\) has distinct real roots if \(r > 3\). Remark that for \(r = 3\), the (double) root is \(p = 2/3\). For \(r > 3\) but very close to 3, it follows from the continuity of \(R\) that the roots are close to 2/3.
Descartes’ rule of signs

Theorem (Descartes’ rule of signs)

Let $p(x) = \sum_{i=0}^{m} a_i x^i$ be a polynomial with real coefficients such that $a_m \neq 0$. Define $v$ to be the number of variations in sign of the sequence of coefficients $a_m, \ldots, a_0$. By ‘variations in sign’ we mean the number of values of $n$ such that the sign of $a_n$ differs from the sign of $a_{n-1}$, as $n$ ranges from $m$ down to 1. Then

- the number of positive real roots of $p(x)$ is $v - 2N$ for some integer $N$ satisfying $0 \leq N \leq \frac{v}{2}$,

- the number of negative roots of $p(x)$ may be obtained by the same method by applying the rule of signs to $p(-x)$. 
Example of use of Descartes’ rule

**Example**

Let

\[ p(x) = x^3 + 3x^2 - x - 3. \]

Coefficients have signs `+ + − −`, i.e., 1 sign change. Thus \( ν = 1 \).

Since \( 0 \leq N \leq 1/2 \), we must have \( N = 0 \). Thus \( ν - 2N = 1 \) and there is exactly one positive real root of \( p(x) \).

To find the negative roots, we examine

\[ p(-x) = -x^3 + 3x^2 + x - 3. \]

Coefficients have signs `− + + −`, i.e., 2 sign changes. Thus \( ν = 2 \) and \( 0 \leq N \leq 2/2 = 1 \). Thus, there are two possible solutions, \( N = 0 \) and \( N = 1 \), and two possible values of \( ν - 2N \). Therefore, there are either two or no negative real roots. Furthermore, note that

\[ p(-1) = (-1)^3 + 3 \cdot (-1)^2 - (-1) - 3 = 0, \]

hence there is at least one negative root. Therefore there must be exactly two.
Back to the logistic map and the polynomial $R$.

We use Descartes’ rule of signs.

- $R$ has signed coefficients $- + -$, so 2 sign changes implying 0 or 2 positive real roots.
- $R(-x)$ has signed coefficients $- - -$, so no negative real roots.
- Since $\Delta > 0$, the roots are real, and thus it follows that both roots are positive.

To show that the roots are also smaller than 1, consider the change of variables $z = x - 1$. The polynomial $R$ is transformed into

$$R_2(z) = -r^3(z + 1)^2 + r^2(r + 1)(z + 1) - r(r + 1)$$

$$= -r^3z^2 + r^2(1 - r)z - r.$$

For $r > 1$, the signed coefficients are $- - -$, so $R_2$ has no root $z > 0$, implying in turn that $R$ has no root $x > 1$. 
If $0 < r < 1$, then $N = 0$ is attracting, $p$ does not exist and there are no period 2 points.

At $r = 1$, there is a bifurcation (called a **transcritical** bifurcation).

If $1 < r < 3$, then $N = 0$ is repelling, $N = p$ is attracting, and there are no period 2 points.

At $r = 3$, there is another bifurcation (called a **period-doubling** bifurcation).

For $r > 3$, both $N = 0$ and $N = p$ are repelling, and there is a period 2 point.
Bifurcation diagram for the discrete logistic map

The logistic map
Bifurcation diagram for the discrete logistic map

The logistic map
This process continues

Bifurcation diagram for the discrete logistic map
The period-doubling cascade to chaos

The logistic map undergoes a sequence of period doubling bifurcations, called the **period-doubling cascade**, as $r$ increases from 3 to 4.

- Every successive bifurcation leads to a doubling of the period.
- The bifurcation points form a sequence, \( \{r_n\} \), that has the property that
  \[
  \lim_{n \to \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n}
  \]
  exists and is a constant, called the Feigenbaum constant, equal to 4.669202...
- This constant has been shown to exist in many of the maps that undergo the same type of cascade of period doubling bifurcations.
After a certain value of $r$, there are periodic points with all periods. In particular, there are periodic points of period 3.

By a theorem (called *Sarkovskii’s theorem*), the presence of period 3 points implies the presence of points of all periods.

At this point, the system is said to be in a chaotic regime, or chaotic.
Bifurcation cascade for $2.9 \leq r \leq 4$
The complete bifurcation cascade

The logistic map
The tent map

[May’s 1976 paper]

\[ X_{t+1} = \begin{cases} 
  aX_t & \text{if } X_t < 1/2 \\
  a(1 - X_t) & \text{if } X_t > 1/2 
\end{cases} \]

defined for \( 0 < X < 1 \).

For \( 0 < a < 1 \), all trajectories are attracted to \( X = 0 \); for \( 1 < a < 2 \), there are infinitely many periodic orbits, along with an uncountable number of aperiodic trajectories, none of which are locally stable. The first odd period cycle appears at \( a = \sqrt{2} \) and all integer periods are represented beyond \( a = (1 + \sqrt{5})/2 \).
Yet another chaotic map

[May’s 1976 paper]

\[ X_{t+1} = \begin{cases} 
\lambda X_t & \text{if } X_t < 1 \\
\lambda X_t^{1-b} & \text{if } X_t > 1 
\end{cases} \]

If \( \lambda > 1 \), GAS point for \( b < 2 \). For \( b > 2 \), chaotic regime with all integer periods present after \( b = 3 \).
The Ricker model

\[ N(t + 1) = N(t) \exp \left\{ r \left( 1 - \frac{N(t)}{K} \right) \right\} = f(N(t)), \]

*r* intrinsic growth rate, *K* carrying capacity. Growth rate *f*(*N*(t)) increasing in *N*(t) and per capita growth \( \frac{f(N)}{N} \) decreasing in *N*(t). Increase in population not sufficient to compensate for decrease in per capita growth, so \( \lim_{N(t) \to +\infty} f(N(t)) = 0 \) (Ricker model is overcompensatory).

- \( r < 2 \) Globally asymptotically stable equilibrium \( \bar{x} = K \)
- \( r = 2 \) Bifurcation into a stable 2-cycle
- \( r = 2.5 \) Bifurcation into a stable 4-cycle
- Series of cycle duplication: 8-cycle, 16-cycle, etc.
- \( r = 2.692 \) chaos
- For \( r > 2.7 \) there are some regions where dynamics returns to a cycle, e.g., \( r = 3.15 \).
Perron-Frobenius theorem

Theorem

If $M$ is a nonnegative primitive matrix, then:

- $M$ has a positive eigenvalue $\lambda_1$ of maximum modulus.
- $\lambda_1$ is a simple root of the characteristic polynomial.
- for every other eigenvalue $\lambda_i$, $\lambda_1 > \lambda_i$ (it is strictly dominant)
- \[
\min \sum_j m_{ij} \leq \lambda_1 \leq \max \sum_j m_{ij}
\]
- \[
\min \sum_i m_{ij} \leq \lambda_1 \leq \max \sum_i m_{ij}
\]
- row and column eigenvectors associated with $\lambda_1$ are $\gg 0$.
- the sequence $M^t$ is asymptotically one-dimensional, its columns converge to the column eigenvector associated with $\lambda_1$; and its rows converges to the row eigenvector associated with $\lambda_1$. 

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