4. Minimal Geršgorin Sets and Their Sharpness

4.1 Minimal Geršgorin Sets

To begin this chapter, recall from Chapter 1 that if \( N := \{1, 2, \ldots, n\} \), if 
\[ A = [a_{i,j}] \in \mathbb{C}^{n \times n} \] 
with \( n \geq 2 \), and if \( \mathbf{x} = [x_1, x_2, \ldots, x_n]^T > 0 \) in \( \mathbb{R}^n \), then 
\[
    r^x_i(A) := \sum_{j \in N \setminus \{i\}} |a_{i,j}|x_j/x_i \quad (i \in N),
\]
\[
    \Gamma^x_i(A) := \{z \in \mathbb{C} : |z - a_{i,i}| \leq r^x_i(A)\} \quad (i \in N), 
\]
and 
\[
    \Gamma^x(A) := \bigcup_{i \in N} \Gamma^x_i(A).
\]

It then follows from Corollary 1.5 that 
\[
    \sigma(A) \subseteq \Gamma^x(A) \quad (\text{any } \mathbf{x} > 0 \text{ in } \mathbb{R}^n),
\]
but as this inclusion holds for any \( \mathbf{x} > 0 \) in \( \mathbb{R}^n \), we immediately have 
\[
    \sigma(A) \subseteq \bigcap_{\mathbf{x} > 0} \Gamma^x(A). \quad (4.1)
\]

The quantity on the right in (4.1) is given a special name in 

**Definition 4.1.** For any \( A = [a_{i,j}] \in \mathbb{C}^{n \times n}, \ n \geq 2 \), then 
\[
    \Gamma^R(A) := \bigcap_{\mathbf{x} > 0} \Gamma^x(A) \quad (4.2)
\]
is the **minimal Geršgorin set for** \( A \), relative to the collection of all weighted row sums, \( r^x(A) \), where \( \mathbf{x} = [x_1, x_2, \ldots, x_n]^T > 0 \) in \( \mathbb{R}^n \).

The minimal Geršgorin set of (4.2) is of interest theoretically because it gives a set, containing \( \sigma(A) \) in the complex plane, which is a subset of the weighted Geršgorin set \( \Gamma^x(A) \) for any \( \mathbf{x} > 0 \) in \( \mathbb{R}^n \).

It is easily seen that the inclusion of (4.1) is valid for each matrix \( B \) in either of the following subsets \( \Omega(A) \) and \( \hat{\Omega}(A) \) of \( \mathbb{C}^{n \times n}, \ n \geq 2, \) where
\[ \Omega(A) := \{B = [b_{i,j}] \in C^{n \times n} : b_{i,i} = a_{i,i} \text{ and } |b_{i,j}| = |a_{i,j}|, i \neq j (\text{all } i, j \in N)\}, \]
and
\[ \hat{\Omega}(A) := \{B = [b_{i,j}] \in C^{n \times n} : b_{i,i} = a_{i,i} \text{ and } |b_{i,j}| \leq |a_{i,j}|, i \neq j (\text{all } i, j \in N)\}, \]
(4.3)

The set \( \Omega(A) \) is called the **equimodular set** and the set \( \hat{\Omega}(A) \) is called the **extended equimodular set** for \( A \). (The sets \( \Omega(A) \) and \( \Omega(A) \) are seen to be more **restrictive** than the equiradial set \( \omega(A) \) and the extended equiradial set \( \hat{\omega}(A) \) of (2.15) and (2.16). See Exercise 2 of this section.) With the notation (cf. (2.17)) of

\[ \sigma(\Omega(A)) := \bigcup_{B \in \Omega(A)} \sigma(B) \text{ and } \sigma(\hat{\Omega}(A)) := \bigcup_{B \in \hat{\Omega}(A)} \sigma(B), \]

it readily follows (see Exercise 3 of this section) that

\[ \sigma(\Omega(A)) \subseteq \sigma(\hat{\Omega}(A)) \subseteq \Gamma^R(A) \]

We now investigate the **sharpness** of the inclusions of (4.5), for any given \( A \in C^{n \times n} \). The following result (which is given as Exercise 4 of this section) is the basis for the definition of the set \( \Omega(A) \) of (4.3).

**Lemma 4.2.** Given matrices \( A = [a_{i,j}] \in C^{n \times n} \) and \( B = [b_{i,j}] \in C^{n \times n}, n \geq 2 \), then \( r_i^*(B) = r_i^*(A) \), for all \( i \in N \) and all \( x = [x_1, x_2, \cdots, x_n]^\top \geq 0 \) in \( \mathbb{R}^n \), if and only if \( |b_{i,j}| = |a_{i,j}| \) for all \( i \neq j \) (\( i, j \in N \)).

Because of Lemma 4.2, we see that, for a given matrix \( A = [a_{i,j}] \in C^{n \times n}, n \geq 2 \), we can only change each nonzero off-diagonal entry by a multiplicative factor \( e^{i\theta} \), to define the associated entry of a matrix \( B = [b_{i,j}] \) in \( \Omega(A) \) of (4.3). This greater restriction gives rise to tighter eigenvalue inclusion results, but the tools for analyzing the matrices in \( \Omega(A) \), as we shall see below, shift from graph theory and cycles of directed graphs, to the Perron-Frobenius theory of nonnegative matrices, our **second recurring theme** in this book.

To start, given any matrix \( A = [a_{i,j}] \in C^{n \times n} \), let \( z \) be any complex number and define the matrix \( Q = [q_{i,j}] \in R^{n \times n} \) by

\[ q_{i,i} := -|z - a_{i,i}| \text{ and } q_{i,j} := |a_{i,j}| \] for \( i \neq j \) \quad (i, j \in N),

so that \( Q \) is dependent on \( A \) and \( z \). It is evident from (4.6) that all off-diagonal entries of \( Q \) are nonnegative real numbers. On setting

\[ \mu := \max \{|z - a_{i,i}| : i \in N\}, \]

and on defining the matrix \( B = [b_{i,j}] \in \mathbb{R}^{n \times n} \) by

\[ b_{i,i} := \mu - |z - a_{i,i}| \text{ and } b_{i,j} := |a_{i,j}| \] for \( i \neq j \) \quad (i, j \in N),
then $B$ has only nonnegative entries, which we write as $B \geq 0$. With the above definitions of $\mu$ and $B$, the matrix $Q$ can then be expressed as

$$Q = -\mu I_n + B,$$

so that the eigenvalues of $Q$ are a simple shift, by $-\mu$, of the eigenvalues of $B$. (The matrix $Q$ is said to be an **essentially nonnegative matrix**, in the terminology of Appendix C, so that, from (C.3), $-Q \in \mathbb{Z}^{n \times n}$; see also Exercise 5 of this section.) Consequently, using the Perron-Frobenius theory of nonnegative matrices (see Theorem C.2 of Appendix C), $Q$ possesses a real eigenvalue $\nu(z)$ which has the property that if $\lambda$ is any eigenvalue of $Q(z)$,

$$\text{Re } \lambda \leq \nu(z).$$

It is also known (from Theorem C.2 of Appendix C) that to $\nu(z)$, there corresponds a nonnegative eigenvector $y$ in $\mathbb{R}^n$, i.e.,

$$Qy = \nu(z)y \text{ where } y \geq 0 \text{ with } y \neq 0,$$

and that $\nu(z)$ can be characterized by

$$\nu(z) = \inf_{x > 0} \left\{ \max_{i \in \mathbb{N}} \left[ \frac{(Qx)_i}{x_i} \right] \right\}.$$

We also note from (4.6) that

$$\frac{(Qx)_i}{x_i} = r^x_i(A) - |z - a_{i,i}| \quad (i \in \mathbb{N}, \: x > 0 \text{ in } \mathbb{R}^n),$$

which will be used below. We further remark that since the entries of the matrix $Q$ are, from (4.6), continuous in the variable $z$, then $\nu(z)$, as an eigenvalue of $Q$, is also a **continuous** function of the complex variable $z$.

The coupling of the function $\nu(z)$ to points of the minimal Geršgorin set $\Gamma^R(A)$ of Definition 4.1 is provided by

**Proposition 4.3.** For any $A = [a_{i,j}] \in \mathbb{C}^{n \times n}, n \geq 2$, then (cf. (4.2))

$$z \in \Gamma^R(A) \text{ if and only if } \nu(z) \geq 0.$$

**Proof.** Suppose that $z$ is an arbitrary point of the minimal Geršgorin set $\Gamma^R(A)$ for $A$. By Definition 4.1, $z \in \Gamma^R(A)$ for each $x > 0$ in $\mathbb{R}^n$. Consequently, for each $x > 0$, there exists an $i \in \mathbb{N}$ (with $i$ dependent on $x$) such that $|z - a_{i,i}| \leq r^x_i(A)$, or equivalently, $r^x_i(A) - |z - a_{i,i}| \geq 0$. But from (4.13), this means, for this value of $i$, that $(Qx)_i/x_i \geq 0$. Hence, $\max_{j \in \mathbb{N}} [(Qx)_j/x_j] \geq 0$ for each $x > 0$, and from (4.12), it follows that $\nu(z) \geq 0$.

Conversely, suppose that $\nu(z) \geq 0$. Then for each $x > 0$ in $\mathbb{R}^n$, (4.12) gives us that there is an $i \in \mathbb{N}$ (with $i$ dependent on $x$) such that
0 \leq \nu(z) \leq (Qx)_i/x_i = r^x_i(A) - |z - a_{i,i}|,

the last equality coming from (4.13). As the above inequalities imply that 
|z - a_{i,i}| \leq r^x_i(A), then \( z \in \Gamma^r_i(A) \), and thus, \( z \in \Gamma^r x(A) \). But as this
inclusion holds for each \( x > 0 \), then \( z \in \Gamma^r(A) \) from (4.2) of Definition 4.1.

With \( \mathbb{C}_\infty := \mathbb{C} \cup \{ \infty \} \) again denoting the extended complex plane, then
\((\Gamma^r(A))' := \mathbb{C}_\infty \setminus \Gamma^r(A)\) denotes the complement of \( \Gamma^r(A) \) in the extended
complex plane \( \mathbb{C}_\infty \). As \( \Gamma^r(A) \) is a compact set in \( \mathbb{C} \), its complement is open
and unbounded. Moreover, Proposition 4.3 shows that \( z \in (\Gamma^r(A))' \) if and
only if \( \nu(z) < 0 \). Now, the boundary of \( \Gamma^r(A) \), denoted by \( \partial \Gamma^r(A) \), is defined
as usual by

\[
\partial \Gamma^r(A) := \overline{\Gamma^r(A)} \cap (\overline{\Gamma^r(A)})' = \Gamma^r(A) \cap (\Gamma^r(A))^c,
\]

the last equality arising from the fact that \( \Gamma^r(A) \) is closed. Thus, it follows
from Proposition 4.3 and the continuity of \( \nu(z) \), as a function of \( z \),
that \( z \in \partial \Gamma^r(A) \) if and only if

\[
\nu(z) = 0, \quad \text{and} \quad \text{there exists a sequence of complex numbers } \{z_j\}_{j=1}^{\infty} \text{ with } \lim_{j \to \infty} z_j = z, \text{ for which } \nu(z_j) < 0 \text{ for all } j \geq 1.
\]

(4.16)

We remark that \( \nu(z) = 0 \) in (4.16) alone does not in general imply that \( z \in \partial \Gamma^r(A) \); see Exercise 9 of this section.

As a first step in assessing the sharpness of the inclusions in (4.5), we
establish

**Theorem 4.4.** For any \( A = [a_{i,j}] \in \mathbb{C}^{n \times n} \) and any \( z \in \mathbb{C} \) with \( \nu(z) = 0 \),
there is a matrix \( B = [b_{i,j}] \in \Omega(A) \) (cf. (4.3)) for which \( z \) is an eigenvalue
of \( B \). In particular, each point of \( \partial \Gamma^r(A) \), from (4.16), is in \( \sigma(\Omega(A)) \), and

\[
\partial \Gamma^r(A) \subseteq \sigma(\Omega(A)) \subseteq \sigma(\hat{\Omega}(A)) \subseteq \Gamma^r(A).
\]

(4.17)

**Proof.** If \( z \in \mathbb{C} \) is such that \( \nu(z) = 0 \), then from (4.11) there is a vector \( y \geq 0 \)
in \( \mathbb{R}^n \) with \( y \neq 0 \), such that \( Qy = 0 \), or equivalently, from (4.6),

\[
\sum_{j \in N \setminus \{k\}} |a_{k,j}|y_j = |z - a_{k,k}|y_k \quad (\text{all } k \in N).
\]

(4.18)

Next, let the real numbers \( \{\psi_j\}_{j=1}^{n} \) satisfy

\[
z - a_{k,k} = |z - a_{k,k}|e^{i\psi_k} \quad (\text{all } k \in N).
\]

(4.19)
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With these numbers \( \{\psi_j\}_{j=1}^n \), define the matrix \( B = [b_{k,j}] \) in \( \mathbb{C}^{n \times n} \) by means of

\[
(4.20) \quad b_{k,k} := a_{k,k} \quad \text{and} \quad b_{k,j} := |a_{k,j}| e^{i\psi_k} \quad \text{for} \quad k \neq j \quad (k, j \in \mathbb{N}),
\]

where it follows from (4.3) that \( B \in \Omega(A) \). On computing \( (By)_k \), we find from (4.20) that

\[
(By)_k = \sum_{j \in N} b_{k,j} y_j = a_{k,k} y_k + e^{i\psi_k} \left( \sum_{j \in N \setminus \{k\}} |a_{k,j}| y_j \right) \quad (k \in \mathbb{N}).
\]

But from (4.19), \( a_{k,k} = z - |z - a_{k,k}| e^{i\psi_k} \) for all \( k \in \mathbb{N} \), and substituting this in the above equations gives

\[
(By)_k = z y_k + e^{i\psi_k} \left[ -|z - a_{k,k}| y_k + \sum_{j \in N \setminus \{k\}} |a_{k,j}| y_j \right] \quad (k \in \mathbb{N}).
\]

As the terms in the above brackets are zero from (4.18) for all \( k \in \mathbb{N} \), the above \( n \) equations can be expressed, in matrix form, simply as

\[
By = zy.
\]

As \( y \neq 0 \), then \( z \) is an eigenvalue of the particular matrix \( B \) of (4.20). Since \( B \in \Omega(A) \), this shows that \( \nu(z) = 0 \) implies \( z \in \sigma(\Omega(A)) \). Finally, since each point \( z \) of \( \partial \Gamma^R(A) \) necessarily satisfies \( \nu(z) = 0 \) from (4.16)i, then \( \partial \Gamma^R(A) \subseteq \sigma(\Omega(A)) \) which, with the inclusions of (4.5), gives the final result of (4.17).

It may come as a surprise that the first inclusion in (4.17) of Theorem 4.4 is valid, without having \( A \) irreducible. To understand why this inclusion is valid also for reducible matrices \( A \), suppose that \( A = [a_{i,j}] \) in \( \mathbb{C}^{n \times n} \), \( n \geq 2 \), is reducible, of the form (cf. (1.19))

\[
A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ \hline O & A_{2,2} \end{bmatrix},
\]

where \( A_{1,1} \in \mathbb{C}^{s \times s} \) and \( A_{2,2} \in \mathbb{C}^{(n-s) \times (n-s)} \), with \( 1 \leq s < n \). The form of the matrix \( A \) above implies that

\[
\sigma(A) = \sigma(A_{1,1}) \cup \sigma(A_{2,2}),
\]

so that the submatrix \( A_{1,2} \) has no effect on the eigenvalues of \( A \). However, the entries of \( A_{1,2} \) can affect the weighted row sums \( r_i^x(A) \) for \( 1 \leq i \leq s \), since
\[ r_i^\star(A) = \sum_{j=1, j \neq i}^s |a_{i,j}|x_j/x_i + \sum_{j=s+1}^n |a_{i,j}|x_j/x_i \quad (1 \leq i \leq s). \]

But, by making the positive components \( x_{s+1}, x_{s+2}, \ldots, x_n \) small while simultaneously making the positive components \( x_1, x_2, \ldots, x_s \) large, the last sum above can be made arbitrarily small, where such choices are permitted because the intersection in (4.2), which defines \( \Gamma^R(A) \), is over all \( x > 0 \) in \( \mathbb{R}^n \).

What remains from Theorem 4.4 is to investigate the sharpness of the final inclusion in (4.17). This is completed in

**Theorem 4.5.** For any \( A = [a_{i,j}] \in \mathbb{C}^{n \times n} \), then (cf. (4.3))

\[ (4.21) \quad \sigma(\hat{\Omega}(A)) = \Gamma^R(A). \]

**Proof.** Let \( z \) be an arbitrary point of \( \Gamma^R(A) \), so that from Proposition 4.3, \( \nu(z) \geq 0 \). From (4.11), there is a vector \( y \geq 0 \) in \( \mathbb{R}^n \) with \( y \neq 0 \), such that \( Qy = \nu(z)y \). The components of this last equation can be expressed (cf. (4.6)) as

\[ (4.22) \quad \sum_{j \in N \setminus \{k\}} |a_{k,j}|y_j = \{|z - a_{k,k}| + \nu(z)\}y_k \quad (\text{all } k \in N). \]

Now, define the matrix \( B = [b_{i,j}] \) in \( \mathbb{C}^{n \times n} \) by

\[ (4.23) \quad b_{k,k} := a_{k,k} \text{ and } b_{k,j} := \mu_k a_{k,j} \text{ for } k \neq j \quad (k, j \in N), \]

where

\[ (4.24) \quad \begin{cases} 
\mu_k := \left( \sum_{j \in N \setminus \{k\}} |a_{k,j}|y_j \right) - \nu(z)y_k, & \text{if } \sum_{j \in N \setminus \{k\}} |a_{k,j}|y_j > 0 \text{, and} \\
\mu_k := 1, & \text{if } \sum_{j \in N \setminus \{k\}} |a_{k,j}|y_j = 0.
\end{cases} \]

From (4.22), (4.24), and the fact that both \( |z - a_{k,k}|y_k \geq 0 \) and \( \nu(z)y_k \geq 0 \) hold for all \( k \in N \), it readily follows that \( 0 \leq \mu_k \leq 1 \) (\( k \in N \)). Thus, from the definitions in (4.3) and (4.23), we see that \( B \in \hat{\Omega}(A) \). Next, on carefully considering the two cases of (4.24), it can be verified, using equations (4.22)-(4.24), that
\[ |z - b_{k,k}|y_k = |z - a_{k,k}|y_k = \left( \sum_{j \in \mathbb{N} \setminus \{k\}} |a_{k,j}|y_j \right) - \nu(z)y_k \]

\[ = \mu_k \cdot \sum_{j \in \mathbb{N} \setminus \{k\}} |a_{k,j}|y_j = \sum_{j \in \mathbb{N} \setminus \{k\}} |b_{k,j}|y_j \quad (k \in \mathbb{N}), \]
i.e.,

\[ |z - b_{k,k}|y_k = \sum_{j \in \mathbb{N} \setminus \{k\}} |b_{k,j}|y_j \quad (k \in \mathbb{N}). \]

Now, the above expression is exactly of the form of that in (4.18) in the proof of Theorem 4.4, and the same proof, as in Theorem 4.4, shows that there is a matrix \( E = [e_{i,j}] \) in \( \mathbb{C}^{n \times n} \), with \( E \in \Omega(B) \), such that \( z \in \sigma(E) \). But, as \( E \in \Omega(B) \) and \( B \in \hat{\Omega}(A) \) together imply (cf. (4.3)) that \( E \in \hat{\Omega}(A) \), then \( z \in \sigma(\Omega(A)) \). \( \blacksquare \)

Theorem 4.5 states that \( \sigma(\hat{\Omega}(A)) \) completely fills out \( \Gamma^{\mathbb{R}}(A) \), i.e., \( \sigma(\hat{\Omega}(A)) = \Gamma^{\mathbb{R}}(A) \), and, as \( \Omega(A) \subseteq \Omega(A) \) from (4.3), this then draws our attention to determining exactly what the spectrum \( \sigma(\Omega(A)) \) actually is. (Note from (4.3) that \( \Omega(A) \) is always a proper subset of \( \hat{\Omega}(A) \), unless \( A \) is a diagonal matix.) While we know from Theorem 4.4 that

\[ \partial \Gamma^{\mathbb{R}}(A) \subseteq \sigma(\Omega(A)) \subseteq \Gamma^{\mathbb{R}}(A) \]

for any \( A \in \mathbb{C}^{n \times n} \), this implies that if \( \sigma(\Omega(A)) \) is a proper subset of \( \Gamma^{\mathbb{R}}(A) \) with \( \sigma(\Omega(A)) \neq \partial \Gamma^{\mathbb{R}}(A) \), then \( \sigma(\Omega(A)) \) necessarily has internal boundaries in \( \Gamma^{\mathbb{R}}(A) \)!

While internal boundaries of \( \sigma(\Omega(A)) \) are discussed in the next section, we include the following result concerning other geometric properties of \( \Gamma^{\mathbb{R}}(A) \).

**Theorem 4.6.** For any irreducible \( A = [a_{i,j}] \in \mathbb{C}^{n \times n} \), \( n \geq 2 \), then \( \nu(a_{i,i}) > 0 \) for each \( i \in \mathbb{N} \). Moreover, for each \( a_{i,i} \) and for each real \( \theta \) with \( 0 \leq \theta \leq 2\pi \), let \( \hat{\rho}_i(\theta) > 0 \) be the smallest \( \rho > 0 \) for which

\[ \nu(a_{i,i} + \hat{\rho}_i(\theta)e^{i\theta}) = 0, \text{ and} \]

\[ \text{there exists a sequence of complex numbers } \{z_j\}_{j=1}^{\infty} \text{ with} \]

\[ \lim_{j \to \infty} z_j = a_{i,i} + \hat{\rho}_i(\theta)e^{i\theta}, \text{ such that } \nu(z_j) < 0 \text{ for all } j \geq 1. \]

Then, the complex interval \([a_{i,i} + te^{i\theta}], \text{ for } 0 \leq t \leq \hat{\rho}_i(\theta)\), is contained in \( \Gamma^{\mathbb{R}}(A) \) for each real \( \theta \), and, consequently, the set

\[ \bigcup_{\theta \text{ real}} [a_{i,i} + te^{i\theta}]_{t=0}^{\hat{\rho}_i(\theta)} \]

is a star-shaped subset (with respect to \( a_{i,i} \)) of \( \Gamma^{\mathbb{R}}(A) \), for each \( i \in \mathbb{N} \).
Proof. First, the irreducibility of $A$, as $n \geq 2$, gives that the matrix $Q$ of (4.6) is also irreducible for any choice of $z$, and this irreducibility also gives (from ii) of Theorem C.1 of Appendix C) the following extended characterization (cf. (4.12)) of $\nu(z)$: for any $z \in \mathbb{C}$, there is a $y > 0$ in $\mathbb{R}^n$ such that

$$\nu(z) = [(Qy)_j/y_j] \quad \text{for all } j \in N.$$ 

Then for any $i \in N$, choose $z = a_{i,i}$ and let $x > 0$ be the associated vector for which the above characterization of $\nu(a_{i,i})$ is valid. Then with $j = i$ in the above display and with (4.13), we have

$$\nu(a_{i,i}) = [(Qx)_i/x_i] = r^x_i(A) - |z - a_{i,i}| = r^x_i(A) > 0,$$

the last inequality holding since $A$ is irreducible. Thus, $\nu(a_{i,i}) > 0$.

Next, for each fixed $\theta$ with $0 \leq \theta \leq 2\pi$, consider the semi-infinite complex line $a_{i,i} + te^{i\theta}$ for all $t \geq 0$, which emanates from $a_{i,i}$. Clearly, the function $\nu(a_{i,i} + te^{i\theta})$ is positive at $t = 0$, is continuous on this line, and is negative, from (4.14), outside the compact set $\Gamma^R(A)$. Thus, there is a smallest $\hat{\rho}_i(\theta) > 0$ which satisfies (4.25), and from (4.16), we see that $(a_{i,i} + \hat{\rho}_i(\theta)e^{i\theta}) \in \partial\Gamma^R(A)$. This means that, for each real $\theta$, the line segment, which joins the point $a_{i,i}$ to the point $a_{i,i} + \hat{\rho}_i(\theta)e^{i\theta}$ on $\partial\Gamma^R(A)$, is a subset of $\Gamma^R(A)$, as is the union (4.26) of all such line segments. For this reason, this set is star-shaped with respect to $a_{i,i}$.

We remark that star-shaped sets are a highly useful concept in function theory, and examples of the star-shaped subsets of Theorem 4.6 will be given later in this section.

Continuing now our discussion of possible internal boundaries of $\sigma(\Omega(A))$ in $\Gamma^R(A)$, consider the irreducible matrix

$$B_3 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \in \mathbb{R}^{3 \times 3}.$$  (4.27)

Then, any matrix $C_3$ in $\Omega(B_3)$ can be expressed, from (4.3), as

$$C_3 = \begin{bmatrix} 2 & 0 & e^{i\psi_1} \\ 0 & 1 & e^{i\psi_2} \\ e^{i\psi_3} & e^{i\psi_4} & 2 \end{bmatrix},$$  (4.28)

where $\{\psi_i\}_{i=1}^4$ are arbitrary real numbers satisfying $0 \leq \psi_i \leq 2\pi$. This suggests the following numerical experiment. Using the random number generator from Matlab 6 to generate a certain number $S$ of strings $\{\beta_j(k)\}_{j=1, k=1}^{4, S}$
of four successive random numbers for the interval $[0,1]$, then the numbers $\{\psi_j(k) := 2\pi \beta_j(k)\}_{j=1}^{S}$ determine $S$ matrices $C_3$ in $\Omega(B_3)$. With $S = 1,000$, the subroutine “eig” of Matlab 6 then numerically determined and plotted all 3,000 eigenvalues of these 1,000 matrices $C_3$, and this is shown in Fig. 4.1, where each “+” is an eigenvalue of some one of these 1,000 matrices. Next, note from (4.3) that $\text{diag}(A) = \text{diag}[a_{1,1}, a_{2,2}, \cdots, a_{n,n}]$ is clearly a matrix in $\Omega(A)$, so that $a_{i,i} \in \sigma(\Omega(A))$ for each $i \in N$. In particular, the diagonal entries 1 and 2 of $B_3$ are necessarily points of $\Gamma^R(B_3)$, but as can be seen from Fig. 4.1, these diagonal elements do not appear to be eigenvalues of $\sigma(\Omega(B_3))$. This figure strongly suggests that $\sigma(\Omega(B_3))$ is a proper subset of $\Gamma^R(B_3)$ with $\sigma(\Omega(B_3)) \neq \partial \Gamma^R(B_3)$, and that internal boundaries do occur!

![Fig. 4.1. Eigenvalues of 1000 random matrices $C_3$ of (4.28)](image)

The precise analytical description of $\sigma(\Omega(B_3))$ will be given in (4.64) and (4.65) of Section 4.2.

As a final example in the section, we also consider the matrix $A_n$ in $\mathbb{C}^{n \times n}$, $n \geq 2$, defined by
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\[ A_n := \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}, \]  

(4.29)

where

\[ a_{k,k} := \exp \left( \frac{2\pi i(k-1)}{n} \right) \quad (k \in \mathbb{N}), \quad \text{and} \quad |a_{1,2}a_{2,3}\cdots a_{n,1}| = 1. \]  

(4.30)

Note that \( A_n \) is necessarily irreducible because, by (4.30), \( a_{1,2}, a_{2,3}, \ldots, a_{n,1} \) are all nonzero. For any \( x = [x_1, x_2, \ldots, x_n]^T > 0 \) in \( \mathbb{R}^n \), it is further clear that \( r_x^X(A_n) = |a_i, i+1| \cdot x_{i+1}/x_i \) for \( 1 \leq i < n \), and that \( r_n^X(A_n) = |a_{n,1}|x_1/x_n \), so that, again with the second part of (4.30),

\[ \prod_{j=1}^{n} r_j^X(A_n) = 1 \quad \text{for all} \quad x > 0. \]  

(4.31)

Now, let \( z \) be any complex number such that \( z \in \partial \Gamma^R(A_n) \). Then from (4.16), \( \nu(z) = 0 \), and because \( A_n \) is irreducible, it can be seen from (4.11) and (4.13) that there is a \( y > 0 \) such that \( |z - a_{i,i}| = r_i^X(A_n) \) for all \( i \in \mathbb{N} \). On taking products and using the result of (4.31), we see that

\[ z \in \partial \Gamma^R(A_n) \quad \text{only if} \quad \prod_{i=1}^{n} |z - a_{i,i}| = 1. \]  

(4.32)

Then writing \( z = r(\theta)e^{i\theta} \) and using the explicit definition of \( a_{k,k} \) in (4.30), it can be verified (see Exercise 8 of this section) that the set in (4.32) reduces to

\[ (r(\theta))^n = 2 \cos(n\theta), \quad \text{for} \quad \theta \in [0, 2\pi], \]

which is the boundary of a lemniscate of order \( n \), as described in Section 2.2. For the special case \( n = 4 \), the matrix \( A_4 \) reduces, from (4.29) and (4.30) with \( a_{1,2} = a_{2,3} = a_{3,4} = a_{4,1} = 1 \), to

\[ A_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & i & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -i \end{bmatrix}, \]  

(4.33)

and the resultant lemniscate boundary for \( \partial \Gamma^R(A_4) \) is shown in Fig. 4.2, along with the “usual” Geršgorin set (cf. (1.5)) \( \Gamma(A_4) \) for \( A_4 \), whose boundary is given by the outer circular arcs.

We remark, from the special form of the matrix \( A_n \) in (4.29), that it can be verified that
4.1 Minimal Geršgorin Sets

Fig. 4.2. $\partial \Gamma^R(A_4)$ and $\Gamma(A_4)$ for $A_4$ of (4.33)

$$\det(A_n - \lambda I_n) = \prod_{i=1}^{n} (a_{i,i} - \lambda) - (-1)^{n}a_{1,2}a_{2,3} \cdots a_{n,1},$$

so that if $\lambda$ is any eigenvalue of any $B \in \Omega(A_n)$, it follows from (4.30) that

$$\prod_{i=1}^{n} |a_{i,i} - \lambda| = 1.$$ 

But from (4.32), we know that (4.33) precisely describes the boundary, $\partial \Gamma^R(A)$, of the minimal Geršgorin set $\Gamma^R(A)$. Thus, for the matrix $A_n$ of (4.29),

$$\partial \Gamma^R(A_n) = \sigma(\Omega(A_n)),$$

which means (cf. Fig. 4.2) that while $\Gamma^R(A_n)$ consists of $n$ petals, where each petal has a nonempty interior, the set $\sigma(\Omega(A_n))$ is exactly the boundary of $\Gamma^R(A_n)$, with $\sigma(\Omega(A_n))$ having no intersection with the interior of $\Gamma^R(A_n)$. Also in this example, the star-shaped subset of $\Gamma^R(A_n)$ of Theorem 4.6, for each diagonal entry, is just the entire petal containing that diagonal entry! Note further that each star-shaped subset cannot be continued beyond the point $z = 0$, as $z = 0$ is a limit point of points $z_j$ with $\nu(z_j) < 0$ (cf. (4.16)).

In the next section, we show, using permutations, how the set $\sigma(\Omega(A))$, with its possible internal boundaries, can be precisely represented.
Exercises

1. Show, with (2.15)-(2.16) and (4.3) that
\[ \Omega(A) = \omega(A) \text{ and } \hat{\Omega}(A) = \hat{\omega}(A) \text{ for } n = 2, \]
and that
\[ \Omega(A) \subseteq \omega(A) \text{ and } \hat{\Omega}(A) \subseteq \hat{\omega}(A) \text{ for } n > 2. \]

2. From the definition of (4.3), verify that the inclusions of (4.5) are valid.

3. For any \( A = [a_{i,j}] \in \mathbb{C}^{2 \times 2} \), show (cf.(4.2) and (2.6)) that \( \Gamma^R(A) = \mathcal{K}(A) \), i.e., the minimal Geršgorin set for \( A \) and the Brauer Cassini set for \( A \) coincide for any \( A \in \mathbb{C}^{2 \times 2} \).

4. Give a proof of Lemma 4.2.

5. With the definition in (4.6) of the matrix \( Q \) (which is dependent on \( z \)), then \( -Q \) is an element of \( \mathbb{Z}^{n \times n} \) of (C.3) of Appendix C, so that
\[ -Q = \mu I_n - B \text{ where } B \geq O. \]
Show that \( z \notin \Gamma^R(A) \) if and only if \( -Q \) is a nonsingular \( M \)-matrix, as defined in Appendix C.

6. With the result of the previous exercise, show that \( z \notin \Gamma^R(A) \) implies that every principal minor of \( -Q \) is positive, where a principal minor, of a matrix \( A = [a_{i,j}] \in \mathbb{C}^{n \times n} \), is the determinant of the matrix which results from deleting the same rows and columns from \( A \). (Hint: Apply (A1) of Theorem 4.6 of Berman and Plemmons (1994), p.134.)

7. For a fixed matrix \( A = [a_{i,j}] \in \mathbb{C}^{n \times n} \), we have from (4.12) and (4.13) that
\[ \nu(z) := \inf_{x > 0} \left\{ \max_{i \in N} (r^x_i(A) - |z - a_{i,i}|) \right\} \text{ (any } z \in \mathbb{C}). \]
With this definition of \( \nu(z) \), show that
\[ |\nu(z) - \nu(z')| \leq |z - z'| \text{ (all } z, \text{ all } z' \text{ in } \mathbb{C}), \]
so that \( \nu(z) \) is uniformly continuous on \( \mathbb{C} \). (Hint: Write \( |z - a_{i,i}| = |(z - z') + (z' - a_{i,i})| \), so that \( |z - a_{i,i}| \leq |z - z'| + |z' - a_{i,i}|. \)
8. Verify, with the definition of \( \{a_{k,k}\}_k^n \) in (4.30), that the lemniscate of (4.32) is indeed given by \( (r(\theta))^n = 2\cos(n\theta) \), for \( \theta \in [0, 2\pi] \).

Fig. 4.3. \( \Gamma^R(A) \) for the matrix \( A \) of Exercise 7

9. Consider the matrix \( A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2e^{2\pi i/3} & 1 \\ 1 & 1 & 2e^{-2\pi i/3} \end{bmatrix} \), so that (cf. (4.6))

\[
Q = \begin{bmatrix}
-|z - 2| & 1 & 1 \\
1 & -|z - 2e^{2\pi i/3}| & 1 \\
1 & 1 & -|z - 2e^{-2\pi i/3}| \\
\end{bmatrix}
\]

Show that \( \nu(0) = 0 \), but that \( \nu(z) > 0 \) in every sufficiently small deleted neighborhood of \( z = 0 \). (Hint: Since \( \nu(z) \) is, from (4.11), an eigenvalue of \( Q \) for any complex number \( z \), then \( \nu(z) = 0 \) implies that \( \det Q = 0 \), where \( \det Q = 0 \), for the given matrix \( A \) of this exercise, satisfies

\[
-|z - 2| \cdot |z - 2e^{2\pi i/3}| \cdot |z - 2e^{-2\pi i/3}| + 2 \\
+|z - 2| + |z - 2e^{2\pi i/3}| + |z - 2e^{-2\pi i/3}| = 0.
\]

Then use the fact that \( \{z \in \mathbb{C} : \det Q = 0\} \) consists, from Fig. 4.3, of a Jordan curve, which is \( \partial \Gamma^R(A) \), and the sole point \( z = 0 \) in its interior, and then use Proposition 4.3.)
10. Give an example of an irreducible matrix $A = [a_{i,j}] \in \mathbb{C}^{n \times n}, n \geq 2$, for which one or more of the functions $\{\hat{\rho}_i(\theta)\}_{i=1}^n$ of Theorem 4.6 can be discontinuous on $[0, 2\pi]$. (Hint: Apply the result of Exercise 1 to a nonconvex Brauer Cassini set.)

### 4.2 Minimal Geršgorin Sets via Permutations

The goal of this section is to obtain the precise characterization, in Theorem 4.11, of the set $\sigma(\Omega(A))$ of (4.3) which will then determine any existing internal boundaries of $\sigma(\Omega(A))$ in $\Gamma^R(A)$, as in Fig. 4.1. To obtain such interior boundaries, we make use of the technique, via permutations of Parodi (1952) and Schneider (1954) in Section 3.1, which can give rise to eigenvalue exclusion sets in $\mathbb{C}$. (The coupling of this technique with minimal Geršgorin sets, in this section, comes from Levinger and Varga (1966a).)

As in Section 3.1, given a matrix $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, and given an $x = [x_1, x_2, \ldots, x_n]^T > 0$ in $\mathbb{R}^n$, let $X := \text{diag} [x_1, x_2, \ldots, x_n]$, so that $X$ is a nonsingular diagonal matrix in $\mathbb{R}^{n \times n}$. If $\phi$ is any fixed permutation on the elements of $N := \{1, 2, \ldots, n\}$, let

$$P_\phi := [\delta_{i,\phi(j)}] \in \mathbb{R}^{n \times n}$$

denote its associated permutation matrix. From the definition in (4.3), assume that $B = [b_{i,j}] \in \Omega(A)$, and consider the matrix $M$, defined in analogy with (3.1) by

$$M := (X^{-1}BX - \lambda I_n) P_\phi = [m_{i,j}] \in \mathbb{C}^{n \times n} \quad (\lambda \in \mathbb{C}),$$

whose entries can be verified to be

$$m_{i,j} := (b_{i,\phi(j)}x_{\phi(j)}/x_i) - \lambda \delta_{i,\phi(j)} \quad (i, j \in N).$$

As in Section 3.1, if $\lambda \in \sigma(B)$, then the matrix $M$ of (4.34) is evidently singular and thus cannot be strictly diagonally dominant (cf. Theorem 1.4). Hence, there exists an $i \in N$ for which

$$|m_{i,i}| \leq \sum_{j \in N \setminus \{i\}} |m_{i,j}|.$$  

With the definition of $\Omega(A)$ in (4.3) and with the definitions of (4.35), it can be verified that the inequality of (4.36), for any $B \in \Omega(A)$, can be expressed, in (4.37), in terms of the familiar weighted row sums $r_i^\chi(A)$ of $A$:

$$|a_{i,i} - \lambda| \leq r_i^\chi(A), \quad \text{if } \phi(i) = i,$$

or

$$|a_{i,i} - \lambda| \geq -r_i^\chi(A) + 2|a_{i,\phi(i)}|x_{\phi(i)}/x_i, \quad \text{if } \phi(i) \neq i.$$
The above inequalities are then used, as in Section 3.1, to define the following sets in the complex plane, which are dependent on $\phi$ and $x$:

$$\Gamma_{r,x}(A):=\begin{cases} \{ z \in \mathbb{C} : |z-a_{i,i}| \leq r_{i}(A) \}, & \text{if } \phi(i) = i, \\ \{ z \in \mathbb{C} : |z-a_{i,i}| \geq -r_{i}(A) + 2|a_{i,\phi(i)}|x_{\phi(i)}/x_{i} \}, & \text{if } \phi(i) \neq i, \end{cases}$$

(4.38)

and

$$\Gamma_{r,x}(A):=\bigcup_{i \in N} \Gamma_{r,x}(A).$$

(4.39)

Thus, from (3.5)-(3.7), the set $\Gamma_{r,x}(A)$ is, for $\phi(i) = i$, a closed disk in $\mathbb{C}$, and is, for $\phi(i) \neq i$, either the extended complex $\mathbb{C}_{\infty}$ or the closed exterior of a disk. But the point here is that $\Gamma_{r,x}(A)$ must contain all the eigenvalue of each $B \in \Gamma(A)$, i.e.,

$$\sigma(\Omega(A)) \subseteq \bigcap_{\phi \in \Phi} \Gamma_{r,x}(A).$$

(4.40)

As the left side of the inclusion of (4.40) is independent of the choice of $x > 0$ in $\mathbb{R}^{n}$ and any permutation $\phi$.

As the left side of the inclusion of (4.40) is independent of the choice of $x > 0$, it is immediate that

$$\sigma(\Omega(A)) \subseteq \bigcap_{x>0} \Gamma_{r,x}(A).$$

(4.41)

Then, we define $\Gamma_{r,x}(A)$ in (4.41) as the minimal Geršgorin set, relative to the permutation $\phi$ on $N$ and the weighted row sums $r_{i}(A)$. We remark that the closed set $\Gamma_{r,x}(A)$ can be the extended complex plane $\mathbb{C}_{\infty}$ for certain permutations $\phi \neq I$, where $I$ is the identity permutation. (An example will be given later in this section.) Next, observe that the inclusion in (4.41) holds for each permutation $\phi$. If $\Phi$ denotes the set of all $n!$ permutations on $N = \{1, 2, \ldots, n\}$, then on taking intersections in (4.41) over all permutations $\phi$ in $\Phi$, we further have

$$\sigma(\Omega(A)) \subseteq \bigcap_{\phi \in \Phi} \Gamma_{r,x}(A).$$

(4.42)

Note that the set $\bigcap_{\phi \in \Phi} \Gamma_{r,x}(A)$ is always a compact set in $\mathbb{C}$, since $\Gamma_{r,x}(A)$ is compact.

A natural question, arising from the inclusion above, is if this inclusion of (4.42) is sharp. The main object of this section is to show in Theorem 4.11 below that, through the use of permutations $\phi$, equality actually holds in (4.42), which settles this sharpness question! The importance of this is that permutations $\phi$, for which $\Gamma_{r,x}(A)$ is not the extended complex plane, will help define the set of all internal and external boundaries of $\sigma(\Omega(A))$.  


The pattern of our development here is similar to that of Section 3.1. Our first objective here is to characterize the set $\Gamma_{\phi}^{R}(A)$ of (4.41) for each permutation $\phi$ on $N$, again using the theory on nonnegative matrices.

Fixing $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, and fixing a permutation $\phi$ on $N$, let $z$ be any complex number, and define the real matrix $Q_{\phi} = [q_{i,j}] \in \mathbb{R}^{n \times n}$ by

\begin{equation}
q_{i,j} := (-1)^{\delta_{i,j}} |a_{i,\phi(j)} - z\delta_{i,\phi(j)}| \quad (i, j \in N),
\end{equation}

so that $Q_{\phi}$ is dependent on $z$, $\phi$ and $A$. Of course, if $\phi$ is the identity permutation on $N$, written $\phi = I$, then the matrix $Q_{I}$ defined by (4.43) is just the matrix of (4.6) in Section 4.1. Note that $Q_{\phi}$, from (4.43), is a real matrix with nonnegative off-diagonal entries and nonpositive diagonal entries. (In the terminology again of Appendix C, $Q_{\phi}$ is an essentially nonnegative matrix.) As in Section 4.1, it again follows from Theorem C.2 of Appendix C that $Q_{\phi}$ possesses a real eigenvalue $\nu_{\phi}(z)$ which has the property that if $\lambda$ is any eigenvalue of $Q_{\phi}$, then

$Re \lambda \leq \nu_{\phi}(z)$,

that to $\nu_{\phi}(z)$ there corresponds a nonnegative eigenvector $y \geq \mathbf{0}$ in $\mathbb{R}^{n}$ such that

\begin{equation}
Q_{\phi}y = \nu_{\phi}(z)y \quad \text{where} \quad y \geq \mathbf{0} \quad \text{with} \quad y \neq \mathbf{0},
\end{equation}

and further that

\begin{equation}
\nu_{\phi}(z) = \inf_{u > \mathbf{0}} \left\{ \max_{i \in N} [(Q_{\phi}u)_{i}/u_{i}] \right\}.
\end{equation}

We further note that, from its definition, $\nu_{\phi}(z)$ is also a continuous function of $z$.

The analog of Proposition 4.3 is

**Proposition 4.7.** For any $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, and for any permutation $\phi$ on $N$, then (cf. (4.41))

\begin{equation}
z \in \Gamma_{\phi}^{R}(A) \quad \text{if and only if} \quad \nu_{\phi}(z) \geq 0.
\end{equation}

**Proof.** To begin, it is convenient to define the quantities

\begin{equation}
s_{i,\phi}^{x}(A; z) := r_{i}^{x}(A) - |a_{i,i} - z| \quad \text{if} \quad \phi(i) = i,
\end{equation}

or

\begin{equation}
s_{i,\phi}^{x}(A; z) := |a_{i,i} - z| + r_{i}^{x}(A) - 2|a_{i,\phi(i)}|x_{\phi(i)}/x_{i} \quad \text{if} \quad \phi(i) \neq i,
\end{equation}

so that from (4.38), $\Gamma_{i,\phi}^{x}(A)$ can be described simply by

$$\Gamma_{i,\phi}^{x}(A) := \left\{ z \in \mathbb{C} : s_{i,\phi}^{x}(A; z) \geq 0 \right\} \quad (i \in N).$$

Next, for any $x > \mathbf{0}$ in $\mathbb{R}^{n}$ with $x = [x_{1}, x_{2}, \cdots, x_{n}]^{T}$, define the vector $w = [w_{1}, w_{2}, \cdots, w_{n}]^{T}$ in $\mathbb{R}^{n}$ by means of
(4.48) \[ w_i := x_{\phi(i)} \quad (i \in N), \]

so that \( w > 0 \). Then, it can be verified from (4.43) and (4.47) that, for any \( x > 0 \) in \( \mathbb{R}^n \),

\[
(4.49) \quad s^x_{i,\phi}(A, z) = \frac{x_{\phi(i)}}{x_i} \left[ (Q_{\phi}(w))_i / w_i \right].
\]

To establish (4.46), assume first that \( z \in \Gamma^R_\phi(A) \), so that (cf. (4.41)) \( z \in \Gamma^x_{i,\phi}(A) \) for every \( x > 0 \). But for each \( x > 0 \), this implies that there exists an \( i \) (with \( i \) dependent on \( x \)) such that \( z \in \Gamma^x_{i,\phi}(A) \), i.e., from the above definition of \( \Gamma^x_{i,\phi}(A) \), \( s^x_{i,\phi}(A; z) \geq 0 \). Since \( x > 0 \), then \( x_{\phi(i)}/x_i > 0 \) for all \( i \in N \), and therefore, using (4.49), it follows that \( (Q_{\phi}(w))_i/w_i \geq 0 \). Thus,

\[
(4.50) \quad \max_{j \in N} \left[ (Q_{\phi}(w))_j / w_j \right] \geq 0 \quad \text{for each } x > 0.
\]

Clearly, as \( x > 0 \) runs over all positive vectors in \( \mathbb{R}^n \), so does its corresponding vector \( w > 0 \), defined in (4.48). But with (4.50), the definition in (4.45) gives us that \( \nu_\phi(z) \geq 0 \).

Conversely, assume that \( \nu_\phi(z) \geq 0 \). Again using (4.45) and (4.49), it follows that, for each \( x > 0 \), there is an \( i \in N \) (with \( i \) dependent on \( x \)), such that \( s^x_{i,\phi}(A; z) \geq 0 \). But this implies that \( z \in \Gamma^x_{i,\phi}(A) \), so that from (4.39), \( z \in \Gamma^x_{i,\phi}(A) \) for each \( x > 0 \). Hence, from the definition in (4.41), \( z \in \Gamma^R_\phi(A) \).

It now follows from Proposition 4.7 and the continuity of \( \nu_\phi(z) \), as a function \( z \), that the boundary, \( \partial \Gamma^R_\phi(A) \) of \( \Gamma^R_\phi(A) \), defined by

\[
\partial \Gamma^R_\phi(A) := \overline{\Gamma^R_\phi(A)} \cap \bigcap \left( \Gamma^R_\phi(A)^c \right),
\]

if it is not empty, can be analogously characterized as follows:

\[
(4.51) \quad \begin{cases} 
  \nu_\phi(z) = 0, \text{ and} \\
  \text{i)} \nu_\phi(z) < 0 \text{ for all } j \geq 1 \end{cases}
\]

We note, of course, that the boundary \( \partial \Gamma^R_\phi(A) \) can be empty, as in the case when \( \Gamma^R_\phi(A) = C_\infty \).

As in Theorem 4.4, we now show that if \( \partial \Gamma^R_\phi(A) \neq \emptyset \), each boundary point of \( \Gamma^R_\phi(A) \) is an eigenvalue of some matrix \( B \in \Omega(A) \).
Theorem 4.8. For any $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, and for any permutation $\phi$ on $N$, then for each $z \in \mathbb{C}$ with $\nu_\phi(z) = 0$, there is a matrix $B = [b_{i,j}] \in \Omega(A)$ for which $z$ an eigenvalue of $B$. In particular, if $\partial \Gamma^R_\phi(A) \neq \emptyset$, each point of $\partial \Gamma^R_\phi(A)$ is in $\sigma(\Omega(A))$, and

\begin{align}
\partial \Gamma^R_\phi(A) & \subseteq \sigma(\Omega(A)) \subseteq \Gamma^R_\phi(A). \quad (4.52)
\end{align}

Proof. First, assume that $z$ is any complex number for which $\nu_\phi(z) = 0$. Then (cf. (4.44)), there exists a nonnegative eigenvector $y \geq 0$ in $\mathbb{R}^n$ with $y \neq 0$, such that $Q_\phi y = 0$. Next, let the real numbers $\{\psi_k\}_{k=1}^n$ satisfy

\begin{align}
z - a_{k,k} = |z - a_{k,k}| e^{i\psi_k} \quad (\text{for all } k \in N),
\end{align}

and define the matrix $B = [b_{i,j}] \in \mathbb{C}^{n \times n}$ by

\begin{align}
b_{k,k} := a_{k,k} \quad \text{and} \quad b_{k,j} := |a_{k,j}| \exp\{i[\psi_k + \pi(-1 + \delta_{k,\phi(k)} + \delta_{j,\phi(k)})]\},
\end{align}

for $k \neq j$, where $k, j \in N$. It is evident from (4.53) and the definition in (4.3) that $B \in \Omega(A)$, and if $w_{\phi(j)} := y_j \quad (j \in N)$,

it can be verified (upon considering separately the cases when $\phi(i) = i$ and $\phi(i) \neq i$) that $Q_\phi y = 0$ is equivalent to

\begin{align}
Bw = zw. \quad (4.54)
\end{align}

But since $y \geq 0$ with $y \neq 0$, then $w \neq 0$, and (4.54) thus establishes that $z$ is an eigenvalue of the matrix $B$, which is an element of $\Omega(A)$. Finally, if $\partial \Gamma^R_\phi(A)$ is not empty, then, as each point of $\partial \Gamma^R_\phi(A)$ necessarily satisfies $\nu_\phi(z) = 0$ from (4.51), the inclusions of (4.52) of Theorem 4.8 directly follow. \hfill \blacksquare

Next, given any $B = [b_{i,j}] \in \mathbb{C}^{n \times n}$, it is convenient to define the associated set, $\Omega^\Delta(B)$, as

\begin{align}
\Omega^\Delta(B) := \{ C = [c_{i,j}] \in \mathbb{C}^{n \times n} : |c_{i,j}| = |b_{i,j}| \quad (i, j \in N) \}.
\end{align}

The set $\Omega^\Delta(B)$, called the rotated equimodular set for $A$, is very much like the equimodular set $\Omega(B)$ of (4.3), but it is larger since the diagonal entries of $B$ (as well as the off-diagonal entries of $B$) are allowed to be multiplied by factors of absolute value 1, in passing to elements of $\Omega^\Delta(B)$. This set $\Omega^\Delta(B)$ arises very naturally from the following question: Suppose $B = [b_{i,j}] \in \mathbb{C}^{n \times n}$ is strictly diagonally dominant, i.e. (cf. (1.11)),

\begin{align}
Q(y) = 0.
\end{align}
\begin{equation}
|b_{i,i}| > \sum_{j \in N \setminus \{i\}} |b_{i,j}| \quad \text{(for all } i \in N),
\end{equation}

so that \( B \) is nonsingular from Theorem 1.4. But as (4.56) implies that each \( C = [c_{i,j}] \in \Omega(B) \) is also strictly diagonally dominant, then each \( C \in \Omega(B) \) is evidently nonsingular. It is natural to ask is if the converse of this is true: If each \( C \in \Omega(B) \) is nonsingular, is \( B \) strictly diagonally dominant? The following easy counterexample shows this to be false. Consider

\[
B := \begin{bmatrix}
2 & 3 \\
5 & 6
\end{bmatrix}, \quad \text{so that } C := \begin{bmatrix}
2e^{i\theta_1} & 3e^{i\theta_2} \\
5e^{i\theta_3} & 6e^{i\theta_4}
\end{bmatrix}
\]

represents, for real \( \theta_i \)'s, an arbitrary element of \( \Omega(B) \). Then,

\[
\det C = 12e^{i(\theta_1+\theta_4)} - 15e^{i(\theta_2+\theta_3)} \neq 0 \quad \text{for any choices of real } \theta_j \text{’s.}
\]

Thus, each matrix in \( \Omega(B) \) is nonsingular, while \( B \), by inspection, fails to be strictly diagonally dominant. But surprisingly, Camion and Hoffman (1966) have shown that the converse of the question posed above is, in a broader sense, true! Their result is

**Theorem 4.9.** Let \( B = [b_{i,j}] \in \mathbb{C}^{n \times n} \), and assume that each matrix in \( \Omega(B) \) of (4.55) is nonsingular. Then, there exist a positive diagonal matrix \( X := \text{diag} \{ x_1, x_2, \ldots, x_n \} \) where \( x_i > 0 \) for all \( i \in N \), and a permutation matrix \( P_{\phi} := [\delta_{i,\phi(j)}] \) in \( \mathbb{R}^{n \times n} \), where \( \phi \) is a permutation on \( N \), such that \( BXP_{\phi} \) is strictly diagonally dominant.

As an illustration of Theorem 4.9, it was shown above that the previous matrix \( B \) in \( \mathbb{R}^{2 \times 2} \) is such that each matrix in \( \Omega(B) \) is nonsingular. Then, the choices of \( X = \text{diag}(1, x_2) \), with \( x_2 > 0 \), and \( P_{\phi} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) give that

\[
BXP_{\phi} = \begin{bmatrix}
3x_2 & 2 \\
6x_2 & 5
\end{bmatrix}.
\]

It is evident that the matrix \( BXP_{\phi} \) is strictly diagonally dominant for any \( x_2 \) with \( 2/3 < x_2 < 5/6 \), thereby corroborating the result of Theorem 4.9.

We continue with

**Lemma 4.10.** For any \( A = [a_{i,j}] \in \mathbb{C}^{n \times n} \), then

\begin{equation}
z \notin \sigma(\Omega(A)) \quad \text{if and only if each matrix } \Omega(A - zI_n) \text{ is nonsingular.}
\end{equation}
Proof. It can be verified (see Exercise 1 of this section), from the definitions of the sets $\Omega(A - zI_n)$ and $\Omega(A)$, that each matrix $R$ in $\Omega(A - zI_n)$ can be uniquely expressed as $R = D(B - zI_n)$, where $D := \text{diag} \{ e^{i\psi_1}, e^{i\psi_2}, \ldots, e^{i\psi_n} \}$ with $\{ \psi_j \}_{j=1}^n$ real, and where $B \in \Omega(A)$. Now, $z \notin \sigma(\Omega(A))$ implies that $\det(B - zI_n) \neq 0$ for every $B \in \Omega(A)$. But as $|\det D| = 1$, then $R = D(B - zI_n)$ implies that $\det R = \det D \cdot \det (B - zI_n) \neq 0$ for each $R \in \Omega(A - zI_n)$, and each $R \in \Omega(A - zI_n)$ is thus nonsingular. The converse follows similarly.

With Lemma 4.10 and the Camion-Hoffman Theorem 4.9, we now establish the sought characterization of $\sigma(\Omega(A))$ in terms of the minimal Geršgorin sets $\Gamma_\phi^{\mathcal{R}}(A)$.

**Theorem 4.11.** For any $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, then

$$\sigma(\Omega(A)) = \bigcap_{\phi \in \Phi} \Gamma_\phi^{\mathcal{R}}(A),$$

where $\Phi$ is the set of all permutations $\phi$ on $N$.

**Proof.** Since $\sigma(\Omega(A)) \subseteq \bigcap_{\phi \in \Phi} \Gamma_\phi^{\mathcal{R}}(A)$ from (4.42), then to establish (4.58), it suffices to show that

$$\bigcap_{\phi \in \Phi} \Gamma_\phi^{\mathcal{R}}(A)' = \left( \bigcap_{\phi \in \Phi} \Gamma_\phi^{\mathcal{R}}(A) \right)'.$$ 

Consider any $z \in \mathbb{C}$ with $z \notin \sigma(\Omega(A))$. From Lemma 4.10 and Theorem 4.9, there exist a positive diagonal matrix $X = \text{diag} \{ x_1, x_2, \ldots, x_n \}$, and a permutation $\psi$ on $N$, with its associated $n \times n$ permutation matrix $P_{\psi} = [\delta_{i,\psi(j)}]$, such that the matrix

$$T := (A - zI_n) \cdot X \cdot P_{\psi} =: [t_{i,j}] \in \mathbb{C}^{n \times n},$$

whose entries are given by

$$t_{i,j} := (a_{i,\psi(j)} - z\delta_{i,\psi(j)})x_{\psi(j)} \quad (i, j \in N),$$

is strictly diagonally dominant, i.e.,

$$|t_{i,i}| > \sum_{j \in N \backslash \{i\}} |t_{i,j}| \quad (\text{all } i \in N).$$
On comparing the definition in (4.60) with the definition in (4.43) of the $n \times n$ matrix $Q_\psi$ and on setting $w_j := x\psi(j)$ for all $j \in N$ so that $w := [w_1, w_2, \ldots, w_n]^T > 0$ in $\mathbb{R}^n$, it can be verified that the inequalities in (4.61) can be equivalently expressed as

(4.62) \[ 0 > \left( \sum_{j \in N \setminus \{i\}} |t_{i,j}| \right) - |t_{i,i}| = (Q_\psi w)_i \text{ (all } i \in N). \]

But since $w > 0$, we see that (4.62) gives, from the definition of $\nu_\psi(z)$ in (4.54), that $\nu_\psi(z) < 0$. Hence, from Proposition 4.7, $z \not\in \Gamma^R_\psi(A)$, which further implies that $z \not\in \bigcap_{\phi \in \Phi} \Gamma^R_\phi(A)$. As $z$ was any point not in $\sigma(\Omega(A))$, (4.59) follows.

Given a matrix $A = [a_{i,j}]$ in $\mathbb{C}^{n \times n}$, it may be the case that a particular permutation $\psi$ on $N$ is such that, for each $x > 0$, there is an $i \in N$ for which (cf.(4.38)) $\Gamma^x_{i,\psi}(A) = \mathbb{C}_\infty$, so that (cf.(4.39)) $\Gamma^R_\psi(A) = \mathbb{C}_\infty$. As this holds for all $x > 0$, then (cf.(4.41)) $\Gamma^R_\psi(A) = \mathbb{C}_\infty$. Obviously, such a permutation $\psi$ plays no role at all in defining $\bigcap_{\phi \in \Phi} \Gamma^R_\phi(A)$, so we call such a permutation $\psi$ a trivial permutation for the matrix $A$. It turns out, however, that each nontrivial permutation $\psi$ for the matrix $A$ may determine some boundary of $\bigcap_{\phi \in \Phi} \Gamma^R_\phi(A)$, which, from (4.58) of Theorem 4.11, then helps to delineate the set $\sigma(\Omega(A))$.

To show this more explicitly, consider again the matrix

(4.63) \[ B_3 := \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \in \mathbb{R}^{3 \times 3}. \]

Using the standard notation (see (2.34)) for representing a permutation in terms of its disjoint cycles, consider the permutation $\phi_1 = (1 \ 2)(3)$, i.e., $\phi_1(1) = 2$, $\phi_1(2) = 1$, $\phi_1(3) = 3$. The associated sets $\Gamma^x_{1,\phi_1}(B_3)$ are, from (4.38), given by

$\Gamma^x_{1,\phi_1}(B_3) = \{ z \in \mathbb{C} : 0 \leq |z - 2| + r^x_1(B_3) \}$,

$\Gamma^x_{2,\phi_1}(B_3) = \{ z \in \mathbb{C} : 0 \leq |z - 1| + r^x_2(B_3) \}$,

$\Gamma^x_{3,\phi_1}(B_3) = \{ z \in \mathbb{C} : |z - 2| \leq r^x_3(B_3) \}$,

where each of the first two sets above is clearly the extended complex plane, for any $x > 0$. Thus, the union of the above sets is the extended complex plane, and $\phi_1$ is then a trivial permutation for $B_3$. It can be verified that, of the six permutations of $(1, 2, 3)$, there are only three nontrivial permutations for $B_3$, namely $(13)(2), (1)(23)$, and $I = (1)(2)(3)$, and the associated
minimal Geršgorin sets for these three permutations can be further verified to be

\[
\begin{align*}
\Gamma^R_l(B_3) &= \{ z \in \mathbb{C} : |z - 2|^2 \cdot |z - 1| \leq |z - 1| + |z - 2| \}, \\
\Gamma^R_{(13)(2)}(B_3) &= \{ z \in \mathbb{C} : |z - 2|^2 \cdot |z - 1| \geq |z - 1| - |z - 2| \}, \\
\Gamma^R_{(1)(23)}(B_3) &= \{ z \in \mathbb{C} : |z - 2|^2 \cdot |z - 1| \geq -|z - 1| + |z - 2| \}.
\end{align*}
\]

(4.64)

The boundaries of the above minimal Geršgorin sets are obviously determined by choosing the case of equality in each of the above sets of (4.64). Specifically,

\[
\begin{align*}
\partial \Gamma^R_l(B_3) &= \{ z \in \mathbb{C} : |z - 2|^2 \cdot |z - 1| = |z - 1| + |z - 2| \}, \\
\partial \Gamma^R_{(13)(2)}(B_3) &= \{ z \in \mathbb{C} : |z - 2|^2 \cdot |z - 1| = |z - 1| - |z - 2| \}, \\
\partial \Gamma^R_{(1)(23)}(B_3) &= \{ z \in \mathbb{C} : |z - 2|^2 \cdot |z - 1| = -|z - 1| + |z - 2| \}.
\end{align*}
\]

(4.65)

As a consequence of Theorem 4.11, \(\sigma(\Omega(B_3))\) is the shaded multiply-connected closed set shown in Fig. 4.4. This example shows how the minimal Geršgorin sets \(\Gamma^R_\phi(B_3)\), for nontrivial permutations \(\phi\) of \(B_3\), can give boundaries defining the spectrum \(\sigma(\Omega(B_3))\). Of course, we note that the boundaries, analytically given in (4.65), are what had been suggested from the numerical computations of the eigenvalues of \(C_3\) in Fig. 4.1!

While it is the case that, for the matrix \(B_3\) of (4.63), each of its three nontrivial permutations gave rise to a different boundary of \(\sigma(\Omega(B_3))\), there are matrices \(A\) in \(\mathbb{C}^{n \times n}\), \(n \geq 2\), where two different nontrivial permutations for this matrix give rise to the same boundary of \(\sigma(\Omega(A))\). (See Exercise 6 of this section.)

Finally in this section, we remark that the irreducible matrix \(B_3\), of (4.63), is such that its cycle set is given by \(C(B_3) := (1\ 3) \cup (2\ 3)\). Now, the cycles (1 3) and (2 3) are not permutations on the entire set \{1 2 3\}, but they become permutations on \{1 2 3\} by annexing missing singletons, i.e., \((1\ 3) \mapsto (1\ 3)(2)\) and \((2\ 3) \mapsto (2\ 3)(1)\) are then permutations on \{1 2 3\}. Curiously, we see that these permutations, on \{1, 2, 3\}, plus the identity mapping, arise in (4.64) as the exact set of nontrivial permutations for the matrix \(B_3\). But, as we shall see, this is no surprise.

Given any irreducible matrix \(A = [a_{i,j}] \in \mathbb{C}^{n \times n}\), \(n \geq 2\), let \(C(A)\) be the cycle set of all (strong) cycles \(\gamma\) in its directed graph, \(G(A)\), as discussed in Section 2.2. Given any (possibly empty) collection \(\{\gamma_j\}_{j=1}^m\) of distinct cycles of \(C(A)\), (i.e., for any \(k\) and \(\ell\) with \(k \neq \ell\) where \(1 \leq k, \ell \leq m\), then \(\gamma_k\) and \(\gamma_\ell\) have no common entries), we annex any singletons necessary to form a permutation on \(N := \{1, 2, \ldots, n\}\), and call \(\Phi(A)\) the set of all such permutations on \(N\). Thus, the identity mapping \(I\) is an element of \(\Phi(A)\). Then,
with the results of Exercise 4 and 5 of this section, we have the new result of Theorem 4.12, which exactly characterizes the set of nontrivial permutations for $A$. (Its proof is left as Exercise 7 of this section.)

**Theorem 4.12.** For any irreducible $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, $n \geq 2$, let $\Phi(A)$ be the collection of permutations on $N := \{1, 2, \ldots, n\}$, derived, as defined above, from the cycle set $C(A)$ of $A$. Then, $\Phi(A)$ is the exact set of nontrivial permutations for $A$.

**Remark.** If $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, $n \geq 2$, has all its off-diagonal entries nonzero, it follows from Theorem 4.12 that every permutation on $N$ is a nontrivial permutation for $A$. (See Exercise 8 of this section).

**Exercises**

1. Given any $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, $n \geq 2$, verify, from the definitions in (4.3) and (4.55), that each $R$ in $\Omega(A)$ can be uniquely expressed as $R = DB$, where $D = \text{diag}[e^{i\psi_1}, \ldots, e^{i\psi_n}]$ with $\{\psi_j\}_{j=1}^n$ real, and where $B \in \Omega(A)$, the equimodular set for $A$. 
2. Verify that of the six permutations on the numbers 1, 2, and 3, only the permutations (13)(2), (1)(23) and \( I = (1)(2)(3) \) give rise to nontrivial permutations for the matrix \( B_3 \) of (4.63).

3. Verify the expressions in (4.64) and (4.65).

4. Given a matrix \( A = [a_{i,j}] \in \mathbb{C}^{n \times n}, \ n \geq 2 \), let \( \phi \) be a permutation on \( N \). Then, \( \phi \) is defined to be a **trivial permutation** for \( A \) if, for each \( y > 0 \) in \( \mathbb{R}^n \), there is an \( i \in N \) for which \( \Gamma_{i,\phi}^y(A) = \mathbb{C}_\infty \). Show that \( \phi \) is a trivial permutation for \( A \) if and only if there is an \( i \) in \( N \) for which \( \phi(i) \neq i \) and for which \( a_{i,\phi(i)} = 0 \). (Hint: Use (4.38).)

5. Consider the irreducible matrix \( A_n = [a_{i,j}] \) of (4.29), where \( n \geq 2 \), where \( |a_{1,2} \cdot a_{2,3} \cdots a_{n,1}| = 1 \), and where \( \{a_{i,i}\}_{i=1}^n \) are arbitrary but fixed complex numbers. Let \( \psi \) be the permutation on \( N \), which, in cyclic notation, is \( \psi = (1 \ 2 \ \cdots \ n) \), i.e.,

\[
\psi(i) = i + 1 \quad (1 \leq i < n), \text{ and } \psi(n) = 1.
\]

If \( \phi \) is a permutation on \( N \) with \( \phi \neq \psi \) and with \( \phi \neq I \), show, using the result of the previous exercise, that \( \Gamma_{\phi}^x(A_n) = \mathbb{C}_\infty \) for any \( x > 0 \) in \( \mathbb{R}^n \), i.e., each permutation \( \phi \), with \( \phi \neq \psi \) and \( \phi \neq I \), is a trivial permutation for \( A_n \).

6. With the definitions of the previous exercise, show (cf. (4.39)) that

\[
\Gamma_I^R(A_n) = \{ z \in \mathbb{C} : \prod_{i=1}^n |z - a_{i,i}| \leq 1 \}, \text{ and }
\]

\[
\Gamma_\psi^R(A_n) = \{ z \in \mathbb{C} : \prod_{i=1}^n |z - a_{i,i}| \geq 1 \},
\]

so that

\[
\Gamma_I^R(A_n) \cap \Gamma_\psi^R(A_n) = \{ z \in \mathbb{C} : \prod_{i=1}^n |z - a_{i,i}| = 1 \}.
\]

Thus, the permutations \( I \) and \( \psi \) give, respectively, the interior and exterior of the single set

\[
\{ z \in \mathbb{C} : \prod_{i=1}^n |z - a_{i,i}| = 1 \}.
\]

7. Using the results of Exercise 4 and 5 above, prove the result of Theorem 4.12.
8. Using Theorem 4.12, show that if $A = [a_{i,j}] \in \mathbb{C}^{n \times n}, n \geq 2$ has all its off-diagonal entries nonzero, then every permutation on $N$ is a nontrivial permutation for $A$.

9. Let $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$ be irreducible, let $\phi$ be any permutation mapping of $N = \{1, 2, \ldots, n\}$ onto $N$, and let $P_{\phi} := [\delta_{i,\phi(j)}]$ be its associated permutation matrix. Show (Loewy (1971), Lemma 1) that the condition, that $a_{i,i} \neq 0$ for each $i$ with $i \neq \phi(i)$, is sufficient for the product $A \cdot P_{\phi}$ to be irreducible. Also, show by means of a $3 \times 3$ matrix, that this condition is not always necessary.

4.3 A Comparison of Minimal Geršgorin Sets and Brualdi Sets

We have examined, in the previous sections of this chapter, the properties of the minimal Geršgorin set for a given matrix $A = [a_{i,j}]$ in $\mathbb{C}^{n \times n}$, but there is the remaining question of how the minimal Geršgorin set, $\Gamma^{R}(A)$ of (4.2) for the matrix $A = [a_{i,j}]$, which is a compact set in the complex plane, compares with its Brualdi set $B(A)$ of (2.40), also a compact set in the complex plane. To answer this question, we begin with

Lemma 4.13. For any $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, then (cf. (4.3) and (2.70))

\[
\hat{\Omega}(A) \subseteq \hat{\omega}_B(A).
\]

Proof. First, assume that $A$ is irreducible with $n \geq 2$, so that, $r_i(A) > 0$ for all $i \in N$. For any matrix $B = [b_{i,j}]$ in $\hat{\Omega}(A)$, we have from (4.3) that

\[
b_{i,i} = a_{i,i} \quad \text{(all } i \in N), \quad \text{and } 0 \leq |b_{i,j}| \leq |a_{i,j}| \quad \text{(all } i \neq j \text{ in } N).
\]

Define the set

\[
S_i(A) := \{ j \in N : j \neq i \text{ and } |a_{i,j}| > 0 \} \quad (i \in N).
\]

Then, $S_i(A) \neq \emptyset$ for any $i \in N$, since $A$ is irreducible. If there is a $j$ in $S_i(A)$ for which $b_{i,j} = 0$, we note from (4.3) and (1.4) that

\[
r_i(B) < r_i(A).
\]

Then, for a fixed $\epsilon > 0$, replace this zero $(i,j)$-th entry of $B$ by any number having modulus $\epsilon$, and do the same for every $k$ in $S_i(A)$ for which $b_{i,k} = 0$, leaving the remaining entries in this $i$-th row, of $B$, unchanged. On carrying
out this procedure for all rows of the matrix $B$, a matrix $B(\epsilon)$, in $\mathbb{C}^{n \times n}$, is created, whose entries are continuous in the parameter $\epsilon$, and for which the cycle set $C(B(\epsilon))$ of $B(\epsilon)$ is identical with the cycle set $C(A)$ of $A$, for each $\epsilon > 0$. In addition, because of the strict inequality in (4.69) whenever $b_{i,j} = 0$ with $j \in S_i(A)$, it follows, for all $\epsilon > 0$ sufficiently small, that
\[
(4.70) \quad r_i(B(\epsilon)) \leq r_i(A) \quad (\text{all } i \in N).
\]
Hence from (2.61), $B(\epsilon) \in \hat{\omega}_B(A)$ for all sufficiently small $\epsilon > 0$. As such, we see from the definition in (2.69) that $B(0) = B \in \bar{\omega}_B(A)$, and, as this holds for any $B \in \hat{\Omega}(A)$, the inclusion of (4.66) is valid in this irreducible case.

For the case $n = 1$, we see by definition that equality holds in (4.66). Finally, assume that $n \geq 2$ and that $A$ is reducible. Without loss of generality, further assume that $A$ is in the normal reduced form of (2.35). Then, our previous construction, in the irreducible case, can be applied to each irreducible submatrices $R_{i,j}$ of case (2.36i), while the remaining case of (2.36ii), as in the case $n = 1$ above, is immediate, completing the proof. 

This brings us to the following new, but not unexpected, result which extends that of Varga (2001a). Its proof is remarkably simple.

**Theorem 4.14.** For any $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, then
\[
(4.71) \quad \Gamma^R(A) \subseteq \mathcal{B}(A).
\]

**Remark.** The word “minimal” in the minimal Geršgorin set of (4.2) seems to be appropriate!

**Proof.** It is known from (4.21) of Theorem 4.5 that
\[
\Gamma^R(A) = \sigma(\hat{\Omega}(A)),
\]
and as $\hat{\Omega}(A) \subseteq \bar{\omega}_B(A)$ from (4.66) of Lemma 4.13, then $\sigma(\hat{\Omega}(A)) \subseteq \sigma(\bar{\omega}_B(A))$. But as $\sigma(\bar{\omega}_B(A)) = \mathcal{B}(A)$ from (2.71) of Theorem 2.11, then
\[
\Gamma^R(A) \subseteq \mathcal{B}(A),
\]
the desired result of (4.71). 

We remark that equality, of the inclusion in (4.71) of Theorem 4.14, can hold, as the following simple example shows. Consider the matrix
\[
(4.72) \quad A_4 = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & i & 1 & 0 \\
0 & 0 & -1 & 1 \\
1 & 0 & 0 & -i
\end{bmatrix},
\]
which was considered in (4.33). Then, $A_4$ is irreducible, and $C(A_4)$ consists of just one cycle, $\gamma = \{1\ 2\ 3\ 4\}$. Thus from (2.40),
As shown in Section 4.2, \( B(A_4) \) of (4.73) is then the set of four petals in Fig. 4.2, along with all their interior points. It can similarly be verified that \( \Gamma^R(A_4) \) is exactly the same set, i.e.,

\[
\Gamma^R(A_4) = B(A_4),
\]

which gives the case of equality in (4.71).

To conclude this chapter, we note that (4.71) of Theorem 4.14 seems, at first glance, somewhat contradictory to the reverse inclusions of (2.9) of Theorem 2.3 and (2.48) of Theorem 2.9, namely,

\[
B(A) \subseteq K(A) \subseteq \Gamma(A),
\]

which is valid for any \( A = [a_{i,j}] \in \mathbb{C}^{n \times n}, n \geq 2 \). (Note that \( n \geq 2 \) is now needed for the definition of the Brauer Cassini set.) But of course, the sets in (4.75) have no dependence on weighted row sums! With \( r_x^\gamma(A) \) of (1.13), we can define, for any \( x = [x_1, x_2, \cdots, x_n]^T > 0 \) in \( \mathbb{R}^n \), the weighted row sum version of the Brauer Cassini oval of (2.5) as

\[
K_{i,j}^r(A) := \{ z \in \mathbb{C} : |z - a_{i,i}| \cdot |z - a_{j,j}| \leq r_x^\gamma(A) \},
\]

where \( i \neq j \); \( i, j \in \mathbb{N} \), and the weighted row sums version of \( K(A) \) of (2.5) as

\[
K^r(A) := \bigcup_{i,j \in \mathbb{N}} K_{i,j}^r(A).
\]

Similarly, if \( \gamma \in \mathcal{C}(A) \) is any cycle of \( \mathcal{G}(A) \), we define

\[
B_{\gamma}^r(A) := \{ z \in \mathbb{C} : \prod_{i \in \gamma} |z - a_{i,i}| \leq \prod_{i \in \gamma} r_x^\gamma(A) \}
\]

and

\[
B^r(A) := \bigcup_{\gamma \in \mathcal{C}(A)} B_{\gamma}^r(A).
\]

Now, applying the inclusions of (4.75) to the matrix \( X^{-1}AX \) where \( X := \text{diag}[x_1, x_2, \cdots, x_n] \) with \( x > 0 \), it directly follows that

\[
B^r(A) \subseteq K^r(A) \subseteq \Gamma^r(A), \text{ for any } x > 0,
\]

so that if

\[
B^R(A) := \bigcap_{x > 0} B^r(A) \text{ and } K^R(A) := \bigcap_{x > 0} K^r(A),
\]

we have from (4.80) that
(4.82) \[ \mathcal{B}^R(A) \subseteq \mathcal{K}^R(A) \subseteq \mathcal{G}^R(A). \]

On the other hand, we have from (4.71) and (4.79) that
\[ \mathcal{G}^R(X^{-1}AX) \subseteq \mathcal{B}(X^{-1}AX) = \mathcal{B}^x(A) \text{ for any } x > 0. \]

But as is easily verified (see Exercise 1 of this section), \( \mathcal{G}^R(X^{-1}AX) = \mathcal{G}^R(A) \) for any \( x > 0 \), so that
\[ \mathcal{G}^R(A) \subseteq \mathcal{B}^x(A) \text{ for any } x > 0. \]

As this inclusion holds for all \( x > 0 \), then with the notation of (4.81), we have
(4.83) \[ \mathcal{G}^R(A) \subseteq \mathcal{B}^R(A). \]

Thus, on combining (4.82) and (4.83), we immediately have the somewhat surprising result of

**Theorem 4.15.** For any \( A = [a_{i,j}] \in \mathbb{C}^{n \times n}, n \geq 2 \), then
(4.84) \[ \mathcal{G}^R(A) = \mathcal{B}^R(A) = \mathcal{K}^R(A). \]

**Exercises**

1. Given \( A = [a_{i,j}] \in \mathbb{C}^{n \times n}, n \geq 2 \), let \( x = [x_1, x_2, \ldots, x_n]^T > 0 \) in \( \mathbb{R}^n \) be any positive vector, and set \( X := \text{diag}[x_1, x_2, \ldots, x_n] \). Show, using (4.2) of Definition 4.1, that
\[ \mathcal{G}^R(X^{-1}AX) = \mathcal{G}^R(A). \]

2. Complete the proof of Lemma 4.13 in the reducible case.
Bibliography and Discussion

4.1 Minimal Geršgorin sets were introduced in Varga (1965), and most of this work is reported in this section. We remark that Theorem 4.7, showing that the spectra of all matrices in $\hat{O}(A)$ of (4.3) exactly fills out the minimal Geršgorin set $\Gamma^R(A)$, was established in Varga (1965). This result was also obtained by Engel (1973) in his Corollary 5.4. Theorem 4.6, having to do with star-shaped subsets associated with diagonal entries of the minimal Geršgorin set $\Gamma^R(A)$, is new. We mention that the term, “minimal Geršgorin circles,” appears in Elsner (1968), but in a quite different context.

It is worth mentioning that the characterization in Section 3.1, of points $z$ in $\Gamma^R(A)$, was through the theory of essentially nonnegative matrices, which are the negatives of matrices in $\mathbb{Z}^{n \times n}$, defined in C.3 of Appendix C. It could, just as well, have been determined from the theory of $M$-matrices because, as is shown in Exercise 3 of Section 4.1, the result that $z \not\in \Gamma^R(A)$ is true if and only if $-Q$ is a nonsingular $M$-matrix.

The exact determination of the minimal Geršgroin set $\Gamma^R(A)$ of a given matrix $A$ in $\mathbb{C}^{n \times n}$ can be computationally challenging, for even relatively low values of $n$. Because of this, their value is more for theoretical purposes.

4.2 The tool for using permutation matrices to obtain new eigenvalue inclusion results, where exteriors of disks then can come into play, is due to Parodi (1952) and Schneider (1954). See also Parodi (1959). The material in this section, based on minimal Geršgorin sets under permutations, comes from Levinger and Varga (1966a), where the aim was to completely specify the boundaries of $\sigma(\Omega(A))$ of (4.4). This was achieved in Theorem 4.11 which was also obtained later by Engel (1973) in his Theorem 5.14. This paper by Engel also contains, in his Corollary 5.11, the result of Camion and Hoffman (1966), mentioned in the proof of our Theorem 4.9.

We note that the exact characterization in Theorem 4.12 of all nontrivial permutations for the matrix $A$, is new.

4.3 The idea of comparing the minimal Geršgorin set, of a given matrix $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, with its Brualdi set in Theorem 4.14 is quite new; see also Varga (2001a). Perhaps the final result of Theorem 4.15 (which also appears in Varga (2001a)) is unexpected.