2. Geršgorin-Type Eigenvalue Inclusion Theorems

2.1 Brauer’s Ovals of Cassini

We begin with the following nonsingularity result of Ostrowski (1937b), where \( r_i(A) \) is defined in (1.4).

**Theorem 2.1.** If \( A = [a_{i,j}] \in \mathbb{C}^{n \times n}, \ n \geq 2, \) and if

\[
|a_{i,i}| \cdot |a_{j,j}| > r_i(A) \cdot r_j(A) \quad \text{(all } i \neq j \text{ in } N := \{1, 2, \ldots, n}\},
\]

then \( A \) is nonsingular.

**Remark.** Note that if \( A \) is strictly diagonally dominant (cf.(1.11)), then (2.1) is valid. Conversely, if (2.1) is valid, then all but at most one of the inequalities of (1.11) must hold. Thus, Theorem 2.1 is a stronger result than Theorem 1.4.

**Proof.** Suppose, on the contrary, that \( A = [a_{i,j}] \in \mathbb{C}^{n \times n} \) satisfies (2.1) and is singular, so that there is an \( x = [x_1, x_2, \ldots, x_n]^T \) in \( \mathbb{C}^n \) with \( x \neq 0 \), such that \( Ax = 0 \). On ordering the components of \( x \) by their absolute values, we can find \( s \) and \( t \) in \( N \) with \( s \neq t \) such that \( |x_t| > 0 \) and

\[
|a_{i,i}| \cdot |x_i| \leq \sum_{j \in N \setminus \{t\}} |a_{i,j}| \cdot |x_j| \quad \text{(all } i \in N\},
\]

(2.3)

On choosing \( i = t \), the inequality of (2.3) becomes, with (2.2),

\[
|a_{t,t}| \cdot |x_t| \leq \sum_{j \in N \setminus \{t\}} |a_{t,j}| \cdot |x_j| \leq r_t(A) \cdot |x_s|.
\]

(2.4)

If \( |x_s| = 0 \), then (2.4) reduces to \( |a_{t,t}| \cdot |x_t| = 0 \), implying that \( |a_{t,t}| = 0 \). But this contradicts the fact, from (2.1), that \( |a_{i,i}| > 0 \) for all \( i \in N \).
Next, assume that $|x_s| > 0$. On choosing $i = s$ in (2.3), we similarly obtain

$$|a_{s,s}| \cdot |x_s| \leq r_s(A) \cdot |x_s|.$$ 

On multiplying the above inequality with that of (2.4), then

$$|a_{t,t}| \cdot |a_{s,s}| \cdot |x_t| \cdot |x_s| \leq r_t(A) \cdot r_s(A) \cdot |x_t| \cdot |x_s|,$$

and as $|x_t| \cdot |x_s| > 0$, this gives $|a_{t,t}| \cdot |a_{s,s}| \leq r_t(A) \cdot r_s(A)$, which contradicts (2.1).

The result of Theorem 2.1 was later rediscovered by Brauer (1947), who used this to deduce the following Geršgorin-type eigenvalue inclusion theorem, which is, by our first recurring theme, equivalent to the result of Theorem 2.1, and hence, needs no proof!

**Theorem 2.2.** For any $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, $n \geq 2$, and any $\lambda \in \sigma(A)$, there is a pair of distinct integers $i$ and $j$ in $N$ such that

$$\lambda \in K_{i,j}(A) := \{z \in \mathbb{C} : |z - a_{i,i}| \cdot |z - a_{j,j}| \leq r_i(A) \cdot r_j(A)\}.$$ 

(2.5)

As this is true for each $\lambda$ in $\sigma(A)$, then

$$\sigma(A) \subseteq K(A) := \bigcup_{i,j \in N \atop i \neq j} K_{i,j}(A).$$ 

(2.6)

The quantity$^1$ $K_{i,j}(A)$, defined in (2.5), is called the $(i,j)$-th **Brauer Cassini oval** for the matrix $A$, while $K(A)$ of (2.6) is called the **Brauer set**. There are now $\binom{n}{2} = \frac{n(n-1)}{2}$ such Cassini ovals for the eigenvalue inclusion of (2.6), as compared with $n$ Geršgorin disks (cf. (1.5)) of the eigenvalue inclusion of (1.7) of Theorem 1.1. Moreover, the compact set $K_{i,j}(A)$ of (2.5) is more complicated than the Geršgorin set, as $K_{i,j}(A)$ can consist of two disjoint components if $|a_{i,i} - a_{j,j}| > 2(r_i(A) \cdot r_j(A))^{1/2}$. (See Exercise 1 of this section.)

To make this clearer, the boundaries of particular Cassini ovals, in a different notation, determined from

$$K(-1, +1, r) := \{z \in \mathbb{C} : |z - 1| \cdot |z + 1| \leq r^2\},$$ 

are shown in Fig. 2.1 for the particular values $r = 0$, $r = 0.5$, $r = 0.9$, $r = 1$, $r = 1.2$, $r = \sqrt{2}$, and $r = 2$. For $r = 0$, this set consists of the two distinct points $z = -1$ and $z = +1$, while for $r = 0.5$, this set consists of two disjoint nearly circular disks which are centered about $-1$ and $+1$. (See Exercise 2 of this section.) For all $r \geq 1$, this set is a closed and connected set in the complex plane $\mathbb{C}$, which is convex for all $r \geq \sqrt{2}$. (See Exercise 4 of this section.)

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$^1$ We use “$K$” here for “Cassini”!
It is interesting to note that $\Gamma(A)$, the union of the $n$ Geršgorin disks in (1.5), and $\mathcal{K}(A)$, the union of the $\binom{n}{2}$ Brauer Cassini ovals in (2.6), both depend solely on the same $2n$ numbers, namely,

\[(2.8) \quad \{a_{i,i}\}_{i=1}^{n} \text{ and } \{r_i(A)\}_{i=1}^{n},\]

derived from the matrix $A$, to obtain the eigenvalue inclusion Theorems 1.1 and 2.2. It is of theoretical interest to ask which of the sets $\Gamma(A)$ and $\mathcal{K}(A)$ is smaller, as the smaller set would give a "tighter" estimate for the spectrum $\sigma(A)$. That $\mathcal{K}(A) \subseteq \Gamma(A)$ holds in all cases is a result, not well known, which was stated by Brauer (1947). As its proof is simple and as the idea of the proof will be used later in Theorem 2.9 of Section 2.3, its proof is given here.

**Theorem 2.3.** For any $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, $n \geq 2$, then (cf. (2.6) and (1.5))

\[(2.9) \quad \mathcal{K}(A) \subseteq \Gamma(A).\]

**Remark.** This establishes that the Brauer set $\mathcal{K}(A)$, for any matrix $A$, is always a subset of its associated Geršgorin set $\Gamma(A)$, but for $n > 3$, there are more, $\binom{n}{2}$, Brauer Cassini ovals to determine, as opposed to the $n$ associated Geršgorin disks.

**Proof.** Let $i$ and $j$ be any distinct integers in $N$ of (2.1), and let $z$ be any point of $K_{i,j}(A)$. Then from (2.5),

\[(2.10) \quad |z - a_{i,i}| \cdot |z - a_{j,j}| \leq r_i(A) \cdot r_j(A).\]
If \( r_i(A) \cdot r_j(A) = 0 \), then \( z = a_{i,i} \) or \( z = a_{j,j} \). But, as \( a_{i,i} \in \Gamma_i(A) \) and \( a_{j,j} \in \Gamma_j(A) \) from (1.5), then \( z \in \Gamma_i(A) \cup \Gamma_j(A) \). If \( r_i(A) \cdot r_j(A) > 0 \), we have from (2.10) that

\[
(2.11) \quad \left( \frac{|z - a_{i,i}|}{r_i(A)} \right) \left( \frac{|z - a_{j,j}|}{r_j(A)} \right) \leq 1.
\]

As the factors on the left in (2.11) cannot both exceed unity, then at least one of these factors is at most unity, i.e., \( z \in \Gamma_i(A) \) or \( z \in \Gamma_j(A) \). Hence, it follows in either case that \( z \in \Gamma_i(A) \cup \Gamma_j(A) \), so that

\[
(2.12) \quad K_{i,j}(A) \subseteq \Gamma_i(A) \cup \Gamma_j(A).
\]

As (2.12) holds for any \( i \) and \( j \) (\( i \neq j \)) in \( N \), we see from (1.5) and (2.6) that

\[
\mathcal{K}(A) := \bigcup_{i,j \in N} K_{i,j}(A) \subseteq \bigcup_{i,j \in N} \{ \Gamma_i(A) \cup \Gamma_j(A) \} = \bigcup_{\ell \in N} \Gamma_{\ell}(A) =: \Gamma(A),
\]

the desired result of (2.9).

We remark that the case of equality in the inclusion of (2.12) is covered (cf. Varga and Krautstengl (1999)) in

\[
(2.13) \quad \begin{cases} K_{i,j}(A) = \Gamma_i(A) \cup \Gamma_j(A) \text{ if and only if} \\ r_i(A) = r_j(A) = 0, \text{ or if } r_i(A) = r_j(A) > 0 \text{ and } a_{i,i} = a_{j,j}. \end{cases}
\]

It is important to remark that while the Brauer set \( \mathcal{K}(A) \) for any \( A = [a_{i,j}] \in \mathbb{C}^{n \times n} \) with \( n \geq 2 \), is always a subset (cf.(2.9)) of the Geršgorin set \( \Gamma(A) \), there is considerably more work, when \( n \) is large, in determining the \( \binom{n}{2} \) Brauer Cassini ovals, than is the case in determining the associated \( n \) Geršgorin disks, i.e., getting the sharper inclusion of (2.9) may come at the price of more computations!

To illustrate the result of (2.9) of Theorem 2.3, consider the \( 4 \times 4 \) irreducible matrix

\[
(2.14) \quad B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1/2 & i & 1/2 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -i \end{bmatrix},
\]

whose row sums \( r_i(B) \) are all unity. The boundary of the Geršgorin set \( \Gamma(B) \) is the outer closed curve in Fig. 2.2, composed of four circular arcs, while the Brauer set \( \mathcal{K}(B) \), from (2.6), is the shaded inner set in Figure 2.2. (The curves in the shaded portion of this figure correspond to internal boundaries of the six Brauer Cassini ovals.) That \( K(B) \subseteq \Gamma(B) \), from (2.9) of Theorem 2.3, is geometrically evident from Fig. 2.2!
As mentioned above, $\Gamma(A)$ and $\mathcal{K}(A)$ depend solely on the same $2n$ numbers of (2.8) which are derived from the matrix $A$, but there is a continuum of matrices (for $n \geq 2$ and some $r_i(A) > 0$) which give rise to the same numbers in (2.8). More precisely, we define the equiradial set for $A$ as

$$\omega(A) := \left\{ B = [b_{i,j}] \in \mathbb{C}^{n \times n} : b_{i,i} = a_{i,i} \text{ and } r_i(B) = r_i(A), \ (i \in N) \right\},$$

(2.15)

and we also define

$$\hat{\omega}(A) := \left\{ B = [b_{i,j}] \in \mathbb{C}^{n \times n} : b_{i,i} = a_{i,i} \text{ and } r_i(B) \leq r_i(A), \ (i \in N) \right\},$$

(2.16)

as the extended equiradial set for $A$, so that $\omega(A) \subseteq \hat{\omega}(A)$. We note, from the final inequality in (2.5), that the eigenvalue inclusion of (2.6) is then valid for all matrices in $\omega(A)$ or $\hat{\omega}(A)$, i.e., with (2.9) and with the definitions of

$$\sigma(\omega(A)) := \bigcup_{B \in \omega(A)} \sigma(B), \text{ and } \sigma(\hat{\omega}(A)) := \bigcup_{B \in \hat{\omega}(A)} \sigma(B),$$

(2.17)
it follows that

\[(2.18) \quad \sigma(\omega(A)) \subseteq \sigma(\hat{\omega}(A)) \subseteq K(A).\]

Again, we are interested in the sharpness of the set inclusions in (2.18), which is covered in Theorem 2.4 below. Engel (1973) obtained (2.19) of Theorem 2.4, which was later independently obtained with essentially the same proof, in Varga and Krautstengl (1999) in the form given below.

**Theorem 2.4.** For any \( A = [a_{i,j}] \in \mathbb{C}^{n \times n}, n \geq 2, \) then

\[(2.19) \quad \sigma(\omega(A)) = \begin{cases} \partial K(A) = \partial K_{1,2}(A) & \text{if } n = 2, \\ K(A) & \text{if } n \geq 3, \end{cases}\]

and, in general, for any \( n \geq 2, \)

\[(2.20) \quad \sigma(\hat{\omega}(A)) = K(A).\]

**Proof.** For \( n = 2, \) each matrix \( B \) in \( \omega(A) \) is, from (2.15), necessarily of the form

\[(2.21) \quad B = \begin{bmatrix} a_{1,1} & r_1(A)e^{i\psi_1} \\ r_2(A)e^{i\psi_2} & a_{2,2} \end{bmatrix}, \text{ with } \psi_1, \psi_2 \text{ arbitrary real numbers.}\]

If \( \lambda \) is any eigenvalue of \( B, \) then \( \det(B - \lambda I) = 0, \) so that from (2.21),

\[(a_{1,1} - \lambda)(a_{2,2} - \lambda) = r_1(A) \cdot r_2(A)e^{i(\psi_1 + \psi_2)}.\]

Hence,

\[(2.22) \quad |a_{1,i} - \lambda| \cdot |a_{2,2} - \lambda| = r_1(A) \cdot r_2(A).\]

As (2.22) corresponds to the case of equality in (2.5), we see that \( \lambda \in \partial K_{1,2}(A). \) Since this is true for any eigenvalue \( \lambda \) of any \( B \) in \( \omega(A) \) and since, from (2.6), \( K_{1,2}(A) = K(A) \) in this case \( n = 2, \) then \( \sigma(\omega(A)) \subseteq \partial K_{1,2}(A) = \partial K(A). \) Moreover, it is easily seen that each point of \( \partial K_{1,2}(A) \) is, for suitable choices of real \( \psi_1 \) and \( \psi_2, \) an eigenvalue of some \( B \) in (2.21), so that \( \sigma(\omega(A)) = \partial K_{1,2}(A) = \partial K(A), \) the desired result of the first part of (2.19).

To establish the second part of (2.19), first assume that \( n \geq 4, \) and consider a matrix \( B = [b_{i,j}] \in \mathbb{C}^{n \times n}, \) which has the partitioned form

\[(2.23) \quad B = \begin{bmatrix} B_{1,1} & B_{1,2} \\ O & B_{2,2} \end{bmatrix}, \]

where

\[(2.24) \quad B_{1,1} := \begin{bmatrix} a_{1,1} & se^{i\psi_1} \\ te^{i\psi_2} & a_{2,2} \end{bmatrix}, \text{ with } 0 \leq s \leq r_1(A), 0 \leq t \leq r_2(A),\]
with \( \psi_1 \) and \( \psi_2 \) arbitrary real numbers, and with \( b_{j,j} = a_{j,j} \) for all \( 1 \leq j \leq n \). Now, for any choices of \( s \) and \( t \) with \( s \in [0, r_1(A)] \) and \( t \in [0, r_2(A)] \), the entries of the block \( B_{1,2} \) can be chosen so that the row sums, \( r_1(B) \) and \( r_2(B) \), in the first two rows of \( B \), equal those of \( A \). Similarly, because \( n \geq 4 \), the row sums of the matrix \( B_{2,2} \) of (2.23) can be chosen to be the same as those in the remaining row sums of \( A \). Thus, by our construction, the matrix \( B \) of (2.23) is an element of \( \omega(A) \). (We remark that this construction fails to work in the case \( n = 3 \), unless \( r_3(A) = 0 \). But from the partitioned form in (2.23), it is evident that

\[
\tag{2.25}
\sigma(B) = \sigma(B_{1,1}) \cup \sigma(B_{2,2}).
\]

Then from the parameters \( s, t, \psi_1, \) and \( \psi_2 \) in \( B_{1,1} \) in (2.24), it can be seen from the definition in (2.5) that for each \( z \in K_{1,2}(A) \), there are choices for these parameters such that \( z \) is an eigenvalue of \( B_{1,1} \). In other words, the eigenvalues of \( B_{1,1} \), on varying the real numbers \( \psi_1 \) and \( \psi_2 \), and \( s \) and \( t \) with \( 0 \leq s \leq r_1(A) \) and \( 0 \leq t \leq r_2(A) \), fill out \( K_{1,2}(A) \), where we note that the remaining eigenvalues of \( B \) (namely, those of \( B_{2,2} \)) must still lie, from (2.18), in \( K(A) \). As this applies to any Cassini oval \( K_{i,j}(A) \) for \( i \neq j \), upon a suitable permutation of the rows and columns of \( B \) of (2.23) which moves row \( i \) into row 1 and row \( j \) into row 2, then \( \sigma(\omega(A)) = K(A) \), for all \( n \geq 4 \).

For the remaining case \( n = 3 \) of (2.19), any matrix \( B \) in \( \omega(A) \) can be expressed as

\[
\tag{2.26}
B = \begin{bmatrix}
a_{1,1} & se^{i\psi_1} & (r_1(A) - s)e^{i\psi_2} \\
te^{i\psi_3} & a_{2,2} & (r_2(A) - t)e^{i\psi_4} \\
u e^{i\psi_5} & (r_3(A) - u)e^{i\psi_6} & a_{3,3}
\end{bmatrix},
\]

where

\[
\tag{2.27}
\begin{align*}
0 \leq s & \leq r_1(A), \quad 0 \leq t \leq r_2(A), \quad \text{and} \quad 0 \leq u \leq r_3(A), \quad \text{and} \\
\{\psi_i\}_{i=1}^6 \quad \text{are arbitrary real numbers.}
\end{align*}
\]

Now, fix any complex number \( z \) in the Brauer Cassini oval \( K_{1,2}(A) \), i.e., (cf. (2.5)), let \( z \) satisfy

\[
\tag{2.28}
|z - a_{1,1}| \cdot |z - a_{2,2}| \leq r_1(A) \cdot r_2(A).
\]

If \( r_1(A) = 0 \), then from (2.28), \( z = a_{1,1} \) or \( z = a_{2,2} \). On the other hand, \( r_1(A) = 0 \) implies that the first row of \( B \) is \([a_{1,1}, 0, 0] \), so that \( a_{1,1} = z \) is an eigenvalue of \( B \). As the same argument also applies to the case \( r_2(A) = 0 \), we assume that \( r_1(A) \cdot r_2(A) > 0 \) in (2.28). Then, let \( s \), with \( 0 \leq s \leq r_1(A) \), be such that \( |z - a_{1,1}| \cdot |z - a_{2,2}| = sr_2(A) \), and select a real number \( \psi \) such that

\[
(a_{1,1} - z) \cdot (a_{2,2} - z) = sr_2(A)e^{i\psi}.
\]

For \( \alpha := r_2(A) + |a_{2,2} - z| \) (so that \( \alpha > 0 \), the matrix \( \tilde{B} \), defined by
\begin{align}
\hat{B} := \begin{bmatrix}
a_{1,1} & se^{i\psi} (r_1(A) - s) \\
r_2(A) & a_{2,2} \\
r_2(A) r_3(A) & \alpha (a_{2,2} - z) r_3(A) \\
\end{bmatrix},
\end{align}

(2.29)

can be verified to be in the set \( \omega(A) \) of (2.15). But a calculation directly shows that \( \det(\hat{B} - zI) = 0 \), so that \( z \) is an eigenvalue of \( \hat{B} \). Hence, each \( z \) in \( K_{1,2}(A) \) is an eigenvalue of some \( B \) in \( \omega(A) \). Consequently, as this construction for \( n = 3 \) can be applied to any point of any Cassini oval \( K_{i,j}(A) \) with \( i \neq j \), then \( \sigma(\omega(A)) = \mathcal{K}(A) \), which completes the proof of (2.19).

The proof of the remaining equality in (2.20) is similar to the above proof, and is left as an exercise. (See Exercise 5 of this section.)

From Theorem 2.4, we remark, for any \( A = [a_{i,j}] \in \mathbb{C}^{n \times n} \) for \( n \geq 3 \), that the Brauer set \( \mathcal{K}(A) \) does a perfect job of estimating the spectra of all matrices in the equiradial set \( \omega(A) \) or in the extended equiradial set \( \hat{\omega}(A) \), which is, in general, not the case for the Geršgorin set \( \Gamma(A) \). In fact, for matrix \( B \) of (2.14), one sees in Fig. 2.2 that there are four unshaded curved domains of \( \Gamma(B) \) which contain no eigenvalue of the set \( \omega(B) \) or \( \hat{\omega}(B) \).

Exercises

1. Show that the Brauer Cassini oval \( K_{i,j}(A), i \neq j \), of (2.5) consists of two disjoint compact sets if \( |a_{i,i} - a_{j,j}| > 2(r_i(A) \cdot r_j(A))^{1/2} \).

2. Consider the level curve \( \{ z \in \mathbb{C} : |z - \alpha| \cdot |z - \beta| = \rho \} \) where \( \beta \neq \alpha \). From Exercise 1, this level curve consists of two disjoint curves, for all \( \rho \) with \( 0 < \rho < |\beta - \alpha|^2/4 \). Show, more precisely, that each of these level curves is nearly a circle, where the radius of each circle is asymptotically \( \frac{\rho}{|\beta - \alpha|} \), as \( \rho \to 0 \).

3. For the Brauer Cassini oval determined from
\[ K(-1, +1, r) := \{ z \in \mathbb{C} : |z - 1| \cdot |z + 1| \leq r^2 \}, \]

show the following statements are valid:

a. \( K(-1, +1, r) \) has two closed components for all \( 0 \leq r < 1 \);

b. \( K(-1, +1, r) \) is a closed and connected set for all \( r \geq 1 \);

c. \( K(-1, +1, r) \) is a convex set for all \( r \geq \sqrt{2} \).

4. Given \( K_{i,j}(A), i \neq j \), of (2.5), show that the result of (2.13) is valid.

5. With the definition of the extended equiradial set \( \hat{\omega}(A) \) of (2.16), establish the result of (2.20) of Theorem 2.4.
6. State and prove an analog of Geršgorin’s Theorem 1.6 (on disjoint subsets of $\Gamma(A)$), for Brauer’s $n(n - 1)/2$ Cassini ovals of an $n \times n$ matrix $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, $n \geq 2$. (Hint: From (2.12), if $\Gamma_i(A)$ and $\Gamma_j(A)$ are disjoint, then $K_{i,j}(A)$ must consist of two disjoint components.)

7. Given any $n \times n$ matrix $A$ with $n \geq 2$, show, as is suggested from Fig. 2.2, that a common point $z$ of the boundaries, $\partial \Gamma_i(A)$ and $\partial \Gamma_j(A)$ ($i \neq j$), of two Geršgorin disks, is a point of $\partial K_{i,j}(A)$.

### 2.2 Higher-Order Lemniscates

Given a matrix $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, let $\{i_j\}_{j=1}^m$ be any $m$ distinct positive integers from $N := \{1, 2, \cdots, n\}$, so that $n \geq m$. Then, the lemniscate\(^2\) of order $m$, derived from $\{i_j\}_{j=1}^m$ and the $2n$ numbers $\{a_{ij}\}_{i=1}^n$ and $\{r_i(A)\}_{i=1}^n$, is the compact set in $\mathbb{C}$ defined by

\[
\ell_{i_1,\ldots,i_m}(A) := \left\{ z \in \mathbb{C} : \prod_{j=1}^m |z - a_{i_j,i_j}| \leq \prod_{j=1}^m r_{i_j}(A) \right\},
\]

and their union, called the lemniscate set for $A$, denoted by

\[
L_{(m)}(A) := \bigcup_{1 \leq i_1, i_2, \ldots, i_m \leq n} \ell_{i_1,\ldots,i_m}(A) \quad (\{i_j\}_{j=1}^m \text{ are distinct in } N),
\]

is over all $\binom{n}{m}$ such choices of $\{i_j\}_{j=1}^m$ from $N$. As special cases, the Geršgorin disks $\Gamma_i(A)$ of (1.5) are lemniscates of order 1, while the Brauer Cassini ovals $K_{i,j}(A)$ of (2.5) are lemniscates of order 2, so that with (1.5) and (2.6)), we have

\[
L_{(1)}(A) = \Gamma(A) \quad \text{and} \quad L_{(2)}(A) = K(A).
\]

As examples of lemniscates, let

\[
\begin{align*}
&\{ a_1 := 1 + i, \ a_2 := -1 + i, \ a_3 := 0, \\
&a_4 := -\frac{1}{4} - 1.4i, \ \text{and} \ a_5 := +\frac{1}{4} - 1.4i,
\end{align*}
\]

and consider the lemniscate boundary of order 5 of

\[
\ell_5(\{a_i\}_{i=1}^5; \rho) := \{ z \in \mathbb{C} : \prod_{i=1}^5 |z - a_i| = \rho \} \quad (\rho \geq 0).
\]

\(^2\) The classical definition of a lemniscate (cf. Walsh (1969), p. 54), is the curve, corresponding to the case of equality in (2.30). The above definition of a lemniscate then is the union of this curve and its interior.
These lemniscate boundaries appear in Fig. 2.3 for the particular values \( \rho = 3, 10, 20, 50 \).

When one considers the proof of Geršgorin’s result (1.7) or the proof of Brauer’s result (2.6), the difference is that the former focuses on \textbf{one} row of the matrix \( A \), while the latter focuses on \textbf{two} distinct rows of the matrix \( A \). But from the result of (2.9) of Theorem 2.3, this would seem to suggest that “using more rows in \( A \) gives better eigenvalue inclusion results for the spectrum of \( A \)”.

Alas, it turns out that \( \mathcal{L}_{(m)}(A) \), as defined in (2.31), \textbf{fails}, in general for \( m > 2 \), to give a set in the complex plane which contains the spectrum of each \( A \) in \( \mathbb{C}^{n \times n}, n \geq m \), as the following example (attributed to Morris Newman in Marcus and Minc (1964), p.149) shows. (See also Horn and Johnson (1985), p.382, for a nice treatment of this example.) It suffices to consider the \( 4 \times 4 \) matrix

\[
A := \begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

where \( \sigma(A) = \{0, 1, 1, 2\} \).
where $a_{i,i} = 1$ for $1 \leq i \leq 4$ and where $r_1(A) = r_2(A) = 1; r_3(A) = r_4(A) = 0$. On choosing $m = 3$ in (2.30), then, for any choice of three distinct integers \{i_1, i_2, i_3\} from \{1, 2, 3, 4\}, the product $r_{i_1}(A) \cdot r_{i_2}(A) \cdot r_{i_3}(A)$ is necessarily zero. Thus, the associated lemniscate in (2.30), for the matrix $A$ of (2.33), always reduces to the set of points $z$ for which $|z - 1|^3 = 0$, so that $z = 1$ is its sole point. Hence, with (2.31), $L_{(3)}(A) = \{1\}$, which fails to contain $\sigma(A)$ in (2.33). (The same argument also gives $L_{(4)}(A) = \{1\}$, and this failure can be extended to all matrices of order $n \geq 3$.)

To obtain a suitable compact set in the complex plane $\mathbb{C}$, based on higher-order lemniscates, which will include all eigenvalues of any given matrix $A$, such as in (2.33), we describe below a modest extension of an important work of Brualdi (1982), which introduced the notion of a cycle\(^3\), from the directed graph of $A$, to obtain an eigenvalue inclusion region for any $A$. This extension is also derived from properties of the directed graph of the matrix $A$.

Given $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, $n \geq 1$, let $\mathbb{G}(A)$ be its directed graph, as described in Section 1.2, on $n$ distinct vertices \{v_1, v_2, \ldots, v_n\}, which consists of a (directed) arc $v_i \rightarrow v_j$, from vertex $v_i$ to vertex $v_j$, only if $a_{i,j} \neq 0$. (This directed graph $\mathbb{G}(A)$ allows loops, as in Section 1.2.) A **strong cycle** $\gamma$ in $\mathbb{G}(A)$ is defined as a sequence $\{i_j\}_{j=1}^{p+1}$ of integers in $N$ such that $p \geq 2$, the elements of $\{i_j\}_{j=1}^p$ are all distinct with $i_{p+1} = i_1$, and $v_{i_1}v_{i_2}, \ldots, v_{i_p}v_{i_{p+1}}$ are arcs of $\mathbb{G}(A)$. This implies that the associated entries of $A$, namely

$$a_{i_1,i_2}, a_{i_2,i_3}, \ldots, a_{i_{p+1},i_1},$$

are all nonzero (where $i_{p+1} = i_1$).

It is convenient to express this strong cycle in standard cyclic permutation notation (cf. Birkhoff and MacLane (1960), p.133)

$$\gamma := (i_1 \ i_2 \ \cdots \ i_p) \quad \text{for} \ \ p \geq 2,$$

where $i_1, i_2, \ldots, i_p$ are distinct integers in $N$, and where $\gamma$ is regarded as the permutation mapping defined by $\gamma(i_1) := i_2, \gamma(i_2) := i_3, \ldots$, and $\gamma(i_p) := i_1$. We also say that this strong cycle $\gamma$ passes through the vertices $\{v_{i_j}\}_{j=1}^p$, and that $\gamma$ has length $p$, with $p \geq 2$. (Note that a loop in $\mathbb{G}(A)$ cannot be a strong cycle since its length would be unity.) If there is a vertex $v_i$ of $\mathbb{G}(A)$ for which there is no strong cycle passing through $v_i$, then we define its associated **weak cycle** $\gamma$ simply as $\gamma = (i)$, independent of whether or not $a_{i,i} = 0$, and we say that $\gamma$ passes through the vertex $v_i$. Next, on defining the **cycle set** $C(A)$ to be the set of all strong and weak cycles of $\mathbb{G}(A)$, then for each vertex $v_i$ of $\mathbb{G}(A)$, there is always a cycle of $C(A)$ which passes through $v_i$. For example, if $n = 1$, then $C(A) = \{1\}$, and there is a unique (weak) cycle through vertex $v_1$.

Continuing, assume that $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, $n \geq 2$, is reducible. From the discussion in Section 1.2, there is a permutation matrix $P \in \mathbb{R}^{n \times n}$ and

\(^3\)More precisely, the word “circuit” is used in Brualdi (1982) for what is called a “cycle” above. Our usage here agrees with that of Horn and Johnson (1985), p.383.
a positive integer $m$, with $2 \leq m \leq n$, such that $PAP^T$ is in the **normal reduced form** of

$$
(2.35) 
\begin{bmatrix}
R_{1,1} & R_{1,2} & \cdots & R_{1,m} \\
O & R_{2,2} & \cdots & R_{2,m} \\
\vdots & \vdots & \ddots & \vdots \\
O & O & \cdots & R_{m,m}
\end{bmatrix},
$$

where each matrix $R_{j,j}$, $1 \leq j \leq m$, in (2.35) is such that

$$
\begin{cases}
  i) & R_{j,j} \text{ is a } p_j \times p_j \text{ irreducible matrix with } p_j \geq 2, \\
  ii) & R_{j,j} \text{ is a } 1 \times 1 \text{ matrix with } R_{j,j} = [a_{k,k}] \text{ for some } k \in \mathbb{N}.
\end{cases}
$$

Of course, if the given matrix $A \in \mathbb{C}^{n \times n}$, $n \geq 2$, is irreducible, we can view this as the case $m = 1$ of (2.35) and (2.36).

Some easy observations follow. The existence of an $R_{j,j} = [a_{k,k}]$ in (2.36ii) is, by definition, equivalent to the statement that vertex $v_k$ of $\mathbb{G}(A)$ has no strong cycle through it. Similarly, the existence of an $R_{j,j}$ satisfying (2.36i) implies that there is at least one strong cycle through each vertex $v_k$ of $\mathbb{G}(A)$, associated with the irreducible submatrix $R_{j,j}$. (See Exercise 2 of this section.) We also note from (2.35) that

$$
\sigma(A) = \bigcup_{k=1}^{m} \sigma(R_{k,k}),
$$

so that the upper triangular blocks $R_{j,k}$ ($j < k \leq m$) in (2.35), if they exist, have no effect on the eigenvalues of $A$. Because of this last observation, we define the new rows sums $\tilde{r}_i(A)$ of $A$ as

$$
(2.37) 
\tilde{r}_i(A) := r_\ell(R_{j,j}),
$$

if the $i$th row of $A$ corresponds to the $\ell$th row of $R_{j,j}$ in (2.35). These new row sums are the old row sums if $A$ is irreducible. Also, we see that (2.37) implies that $\tilde{r}_i(A) = 0$ for any vertex $v_i$ corresponding to a weak cycle in $\mathbb{G}(A)$, (which is consistent with the convention used in (1.4)). Similarly, $\tilde{r}_i(A) > 0$ for each row, corresponding to an irreducible matrix $R_{j,j}$ of (2.36i).

To summarize, these new definitions, of strong and weak cycles and of modified row sums, are all results from a more detailed study of the directed graph $\mathbb{G}(A)$ of $A$.

With the above notations, given any $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, $n \geq 1$, if (cf.(2.34)) $\gamma = (i_1 \ i_2 \ \cdots \ i_p)$, with distinct elements $\{i_j\}_{j=1}^{p}$ and with $p \geq 2$, is a **strong cycle** in $\mathbb{G}(A)$, its associated **Brualdi lemniscate**, $\mathcal{B}_\gamma(A)$, of order $p$, is defined by

$$
(2.38) 
\mathcal{B}_\gamma(A) := \{z \in \mathbb{C} : \prod_{i \in \gamma} |z - a_{i,i}| \leq \prod_{i \in \gamma} \tilde{r}_i(A)\}.
$$
If $\gamma = (i)$ is a weak cycle in $G(A)$, its associated Brualdi lemniscate $B_\gamma(A)$ is defined by

$$B_\gamma(A) := \{ z \in \mathbb{C} : |z - a_{i,i}| = \tilde{r}_i(A) = 0 \} = \{ a_{i,i} \}.$$  

The Brualdi set for $A$ is then defined as

$$B(A) := \bigcup_{\gamma \in C(A)} B_\gamma(A).$$

We now establish our new extension of a result of Brualdi (1982).

**Theorem 2.5.** For any $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$ and any eigenvalue $\lambda$ of $A$, there is a (strong or weak) cycle $\gamma$ in the cycle set $C(A)$ such that (cf. (2.38) or (2.39))

$$\lambda \in B_\gamma(A).$$  

Consequently (cf. (2.40)),

$$\sigma(A) \subseteq B(A).$$

**Proof.** If $n = 1$, then $A = [a_{1,1}] \in \mathbb{C}^{1 \times 1}$, and $C(A)$ consists of the sole weak cycle $\gamma = (1)$. From (2.39), $B_\gamma(A) = \{ a_{1,1} \}$, and the sole eigenvalue of $A$, i.e., $a_{1,1}$, is exactly given by $B_\gamma$. Thus, (2.41) and (2.42) trivially follow. Next, assume that $n \geq 2$. Let $\lambda$ be an eigenvalue of $A$, and assume that $\lambda = a_{k,k}$ for some $k \in N$. Then, there is a (strong or weak) cycle $\gamma$ in $C(A)$ such that $k \in \gamma$. If $\gamma$ is a weak cycle through vertex $v_k$, then $B_\gamma(A) = \{ a_{k,k} \} = \lambda$, so again, (2.41) is satisfied. If $\gamma$ is a strong cycle in $C(A)$ which passes through $v_k$, then with the choice of $z := a_{k,k}$, we see from (2.38) that $a_{k,k} \in B_\gamma(A)$, and thus $\lambda = a_{k,k} \in B(A)$. If each eigenvalue of $A$ is a diagonal entry of $A$, the preceding argument gives $\sigma(A) \subseteq B(A)$, the desired result of (2.42).

Next, assume that $\lambda$ is an eigenvalue of $A$ with $\lambda \neq a_{j,j}$ for any $j \in N$. If $A$ is reducible, it follows from the dichotomy in (2.36) that $\lambda$ must be an eigenvalue of some irreducible matrix $R_{j,j}$, of order $p_j$, with $2 \leq p_j \leq n$. Similarly, if $A$ is irreducible (i.e., the case $m = 1$ of (2.35)), then $\lambda$ is an eigenvalue of the irreducible matrix $R_{1,1} = A$, of order $n$. To simplify notations, we assume that $A = R_{j,j}$ is irreducible. (This means that we will use below the old row sums $r_i(A)$, rather than the new row sums $\tilde{r}_i(A)$ of $R_{j,j}$.) Writing $Ax = \lambda x$, where $x \in \mathbb{C}^n$ with $x = [x_1, x_2, \ldots, x_n]^T \neq 0$, assume that $x_i \neq 0$. Then, $(Ax)_i = \lambda x_i$ gives

$$(\lambda - a_{i,i})x_i = \sum_{j \in N \setminus \{i\}} a_{i,j}x_j.$$  

As $(\lambda - a_{i,i})x_i \neq 0$, all the products $a_{i,j}x_j$ in the above sum cannot be zero. Hence, there is a $k \in N$, with $k \neq i$, such that

$$|x_k| = \max \{|x_j| : j \in N \text{ with } j \neq i \text{ and } a_{i,j}x_j \neq 0\}.$$
Thus, \(|x_k| > 0\) and \(a_{i,k} \neq 0\), so that
\[
|\lambda - a_{i,i}| \cdot |x_i| \leq \sum_{j \in N \setminus \{i\}} |a_{i,j}| \cdot |x_j| \leq r_i(A) \cdot |x_k|,
\]
with \(k \neq i\).

Calling \(i := i_1\) and \(k := i_2\), we can repeat the above process, starting with
\[
(\lambda - a_{i_2,i_2})x_{i_2} = \sum_{j \in N \setminus \{i_2\}} a_{i_2,j}x_j,
\]
and there is similarly an \(i_3\), with \(|x_{i_3}| \neq 0\), and \(a_{i_2,i_3} \neq 0\) such that
\[
|\lambda - a_{i_2,i_2}| \cdot |x_{i_2}| \leq r_{i_2}(A) \cdot |x_{i_3}|,
\]
where \(i_3 \neq i_2\), and where
\[
|x_{i_3}| = \max\{|x_j| : j \in N \text{ with } j \neq i_2 \text{ and } a_{i_2,j}x_j \neq 0\}.
\]

If \(i_3 = i_1\), the process terminates, having produced the distinct integers \(i_1, i_2, i_3\), with \(i_3 = i_1\), and with \(a_{i_1,i_2}\) and \(a_{i_2,i_1}\) nonzero. Thus, with the notation of (2.34), the strong cycle \(\gamma := (i_1 \ i_2)\) has been produced. If \(i_3 \neq i_1\), this process continues, but eventually terminates (since \(N\) is a finite set) when an \(i_{p+1} \in N\) is found which is equal to some previous \(i_\ell\). In either case, there is a sequence \(\{i_j\}_{j=\ell}^p\), with \(p \geq 2\), of distinct integers in \(N\), with \(i_{p+1} = i_\ell\). But, this sequence also produces the following nonzero entries of \(A\):
\[
a_{i_\ell,i_{\ell+1}}, a_{i_{\ell+1}, i_{\ell+2}}, \ldots, a_{i_p,i_{p+1}}, \text{ with } i_{p+1} = i_\ell,
\]
so that, with our notation of (2.34), \(\gamma := (i_\ell \ i_{\ell+1} \ \cdots \ i_p)\) is then a strong cycle of \(A\). Thus, we have that
\[
|\lambda - a_{i_j,i_j}| \cdot |x_{i_j}| \leq r_{i_j}(A) \cdot |x_{i_{j+1}}|, j = \ell, \ell + 1, \ldots, p, \text{ where } i_{p+1} = i_\ell,
\]
where the \(x_{i_j}\)'s are all nonzero. Taking the products of the above gives
\[
\left(\prod_{j=\ell}^p |\lambda - a_{i_j,i_j}|\right) \cdot \left(\prod_{j=\ell}^p |x_{i_j}|\right) \leq \left(\prod_{j=\ell}^p r_{i_j}(A)\right) \cdot \left(\prod_{j=\ell}^p |x_{i_{j+1}}|\right).
\]

But, as \(x_{i_{p+1}} = x_{i_\ell}\), then \(\prod_{j=\ell}^p |x_{i_j}| = \prod_{j=\ell}^p |x_{i_{j+1}}| > 0\), so that, on cancelling these products in the above display, we have
\[
\prod_{j=\ell}^p |\lambda - a_{i_j,i_j}| \leq \prod_{j=\ell}^p r_{i_j}(A).
\]

Thus (cf. (2.38)), \(\lambda \in B_\gamma(A)\), giving (2.41). \(\blacksquare\)
As an application of Theorem 2.5, consider the matrix $B$ of (2.33), which is already in normal reduced form of (2.35). Its cycle set $C(A)$ consists of the strong cycle $\gamma_1 = (1 \ 2)$, and the weak cycles $\gamma_2 = (3)$ and $\gamma_3 = (4)$. It follows from (2.40) that

$$B(B) = \{ z \in \mathbb{C} : |z - 1|^2 \leq 1 \} \cup \{ 1 \} \cup \{ 1 \},$$

which now nicely contains $\sigma(B) = \{0, 1, 1, 2\}$.

We next state the equivalent nonsingularity result associated with Theorem 2.5, which slightly extends the corresponding result in Brualdi (1982).

**Theorem 2.6.** If $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, if $C(A)$ is the set of all strong and weak cycles in $\mathcal{G}(A)$, and if

$$(2.43) \quad \prod_{i \in \gamma} |a_{i,i}| > \prod_{i \in \gamma} \tilde{r}_i(A) \quad (\text{all } \gamma \in C(A)),$$

then $A$ is nonsingular.

Now, we come to an analog, Theorem 2.7, of Taussky’s nonsingularity result of Theorem 1.11, which also appears in Brualdi (1982). (Its proof, which uses the irreducibility of $A$, is left as Exercise 3 of this section, as it follows along the lines of the proofs of Theorems 1.11 and 2.5.)

**Theorem 2.7.** If $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$ is irreducible, and if

$$(2.44) \quad \prod_{i \in \gamma} |a_{i,i}| \geq \prod_{i \in \gamma} r_i(A) \quad (\text{all } \gamma \in C(A)),$$

with strict inequality holding for some $\gamma \in C(A)$, then $A$ is nonsingular.

Then for completeness, the equivalent eigenvalue inclusion result associated with Theorem 2.7 is the following result, which is a generalization of Taussky’s Theorem 1.12.

**Theorem 2.8.** If $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$ is irreducible, and if $\lambda$, an eigenvalue of $A$, is such that $\lambda \not\in \text{int} B_\gamma(A)$ for any $\gamma \in C(A)$, i.e.,

$$(2.45) \quad \prod_{i \in \gamma} |\lambda - a_{i,i}| \geq \prod_{i \in \gamma} r_i(A) \quad (\text{all } \gamma \in C(A)),$$

then $\lambda$ is on the boundary of the Brualdi lemniscate $B_\gamma(A)$, for each $\gamma$ in $C(A)$. In particular, if some eigenvalue $\lambda$ of $A$ lies on the boundary of the Brualdi set $B(A)$ of (2.40), then (2.45) holds.
It is worthwhile to examine again the matrix $B$ of (2.33), which is reducible. Theorem 2.5 gives us (cf. (2.42)) that the spectrum of $B$ can be enclosed in the Brualdi set $\mathcal{B}(A)$, which depends on cycles of different lengths, determined from the directed graph of $B$. This raises the following obvious question: If higher-order lemniscates $L_m(B)$ of (2.31), for a fixed $m$, fail to work for the reducible matrix $B$ of (2.33), can they be successfully applied to arbitrary irreducible matrices? To answer this, consider the matrix

$$D = \begin{bmatrix} 1 & 1 & \epsilon & \epsilon \\ 1 & 1 & 0 & 0 \\ \epsilon & 0 & 1 & 0 \\ \epsilon & 0 & 0 & 1 \end{bmatrix}, \quad \text{with} \ \epsilon > 0,$$

which is obtained by adding some nonzero entries to the matrix of (2.33). The directed graph $\mathcal{G}(D)$ of $D$ is (omitting diagonal loops) given in Fig. 2.4, so that $D$ is irreducible, where its cycle set $C(A)$ consists of the strong cycles

$$\gamma_1 = (1 \ 2), \quad \gamma_2 = (1 \ 3), \quad \text{and} \quad \gamma_3 = (1 \ 4), \quad \text{where} \quad r_1(D) = 1 + 2\epsilon, \quad r_2(D) = 1,$$

$$\text{and} \quad r_3(D) = r_4(D) = \epsilon. \quad \text{If we consider the lemniscate of order 4 for this matrix} \ D, \ \text{we obtain, from (2.31) that}$$

$$L_4(D) = \{ z \in \mathbb{C} : |z - 1|^4 \leq \epsilon^2(1 + 2\epsilon) \},$$

which is the disk $\{ z \in \mathbb{C} : |z - 1| \leq \sqrt[4]{\epsilon}(1 + 2\epsilon)^{1/4} \}$. But, as

$$\sigma(D) = \{ 1 - (1 + 2\epsilon^2)^{1/2}, 1, 1, 1 + (1 + 2\epsilon^2)^{1/2} \},$$

it can be verified (see Exercise 5 of this section) that

$$\sigma(D) \not\subset L_4(D), \quad \text{for any} \ \epsilon > 0.$$
Similarly, it can be verified (see Exercise 5 of this section) that (cf. (2.31))

\[ \sigma(D) \not\subset \mathcal{L}_3(D), \text{ for any } \epsilon > 0. \]

In other words, even with an irreducible matrix, lemniscates of a fixed order \( m \geq 3 \), applied to this irreducible matrix, can fail to capture the spectrum of this matrix, while the Brualdi sets always work!

We note that the eigenvalue inclusion of Theorem 2.5, applied to the matrix \( A = [a_{i,j}] \in \mathbb{C}^{n \times n}, n \geq 1 \), now depends on all the quantities of

\[
\{a_{i,i}\}_{i=1}^n, \quad \{\tilde{r}_i(A)\}_{i=1}^n, \quad \text{and the cycle set } \mathcal{C}(A),
\]

which are derived from the matrix \( A \), its directed graph \( G(A) \), and its normal reduced form (cf. (2.35)). It is of course of interest to see how the Brualdi set \( B(A) \) compares with the Brauer set \( K(A) \). This will be done in the next section. We also ask, in the spirit of Theorem 2.4, if the union of the spectra of all matrices which match the data of (2.47), fills out the Brualdi set \( B(A) \) of (2.47). This will be precisely answered in Section 2.4.

Exercises

1. A matrix \( A = [a_{i,j}] \in \mathbb{C}^{n \times n}, n \geq 2 \), is said to be weakly irreducible (see Brualdi (1982)) if there is a strong cycle through each vertex \( v_i \) of \( G(A) \), the directed graph of \( A \). Show that irreducibility and weak irreducibility are equivalent for all matrices in \( \mathbb{C}^{n \times n} \) if \( n = 2 \) or \( n = 3 \). For \( n \geq 4 \), show that irreducibility implies weak irreducibility, but not conversely in general.

2. Given \( A = [a_{i,j}] \in \mathbb{C}^{n \times n}, n \geq 2 \), show that the existence of an irreducible submatrix \( R_{j,j} \) in (2.36) and (2.35) implies that there is a strong cycle through each vertex of \( G(A) \), associated with the submatrix \( R_{j,j} \).

3. Give a complete proof of Theorem 2.7. (Hint: Follow the general outline of the proofs of Theorem 1.11 and 2.5.)

4. Consider the matrix \( B(\epsilon) := \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \epsilon \\ 0 & 0 & \epsilon & 1 \end{bmatrix} \), where \( \epsilon > 0 \).
   a. Show that the eigenvalues of \( B(\epsilon) \) are \( \{0, 1 - \epsilon, 1 + \epsilon, 2\} \).
   b. Show that \( B(\epsilon) \) is weakly irreducible, in the sense of Exercise 1, for any \( \epsilon > 0 \).
   c. Show that the (strong) cycles of \( B(\epsilon) \) are \( \gamma_1 = (1 \ 2) \) and...
\[ \gamma_2 = (3\ 4), \text{ and that } B_{\gamma_1}(B(\epsilon)) \cup B_{\gamma_2}(B(\epsilon)) \text{ does include all the eigenvalues of } B(\epsilon). \]

5. Consider the matrix \( D \) of (2.46). With the definition of \( L_{(m)}(A) \) in (2.31),
   a. Show that \( \sigma(D) \not\subset L_{(4)}(D) \) for any \( \epsilon > 0 \);
   b. Show that \( \sigma(D) \not\subset L_{(3)}(D) \) for any \( \epsilon > 0 \);
   c. Determine \( B(D) \) from (2.40), and verify that \( \sigma(D) \subseteq B(D) \), for any \( \epsilon > 0 \).

6. Let \( \gamma = (1\ 2\ 3\ 4) \) be a strong cycle from \( C(A) \) for the matrix \( A \). For any of the following five permutations of \( \gamma \), i.e., \( (1\ 2\ 4\ 3), (1\ 3\ 2\ 4), (1\ 3\ 4\ 2), (1\ 4\ 2\ 3) \) and \( (1\ 4\ 3\ 2) \), show that the Brualdi lemniscate set \( B_\gamma(A) \) of (2.38) is unchange. What does this say about the quantities of (2.40)?

7. Consider the irreducible matrix \( E = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix} \), which has an eigenvalue \( \lambda = 0 \), and for which its cycle set \( C(E) \) consists of the two strong cycles \( \gamma_1 = (1\ 2) \) and \( \gamma_2 = (1\ 3) \). Show that not all three Brauer Cassini ovals pass through \( z = 0 \) (Zhang and Gu (1994)). However, show that \( \lambda = 0 \) is a boundary point of its Brualdi set, \( B(E) = B_{\gamma_1}(E) \cup B_{\gamma_2}(E) \), and that \( B_{\gamma_1}(E) \) and \( B_{\gamma_2}(E) \) both pass through \( z = 0 \), as dictated by Theorem 2.8.

8. Let \( A = [a_{i,j}] \in \mathbb{C}^{n \times n} \) be irreducible, with the property that \( A \) has two distinct rows for which each row has two nonzero non-diagonal entries. (This implies that \( n \geq 3 \).) If \( \lambda \in \sigma(A) \) is such that (cf.(2.6)) \( \lambda \in \partial K(A) \), show that \( \lambda \in \partial K_{i,j}(A) \) for all \( i \neq j \), with \( 1 \leq i, j \leq n \). (Rein (1967)). Note that the \( 3 \times 3 \) matrix of Exercise 7 does not satisfy the above hypotheses.

9. The definitions of the Brualdi lemniscates in (2.38) and (2.39) make use of the new row sums \( \{ \tilde{r}_i(A) \}_{i=1}^n \) of (2.37). Show that if these Brualdi lemniscates are defined with the old row sums \( \{ r_i(A) \}_{i=1}^n \), the result of Theorem 2.5 is still valid. (Hint: use (2.49).)
2.3 Comparison of the Brauer Sets and the Brualdi Sets

Our new result here is very much in the spirit of Theorem 2.3.

**Theorem 2.9.** For any $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, $n \geq 2$, then, with the definitions of (2.6) and (2.40),

$$B(A) \subseteq K(A). \tag{2.48}$$

**Remark.** This establishes that the Brualdi set, for any matrix $A$, is always a subset of its associated Brauer set $K(A)$. Note also that the restriction $n \geq 2$ is necessary for the definition of the Brauer set $K(A)$, as in Theorem 2.2.

**Proof.** Consider any cycle $\gamma$ of the cycle set $C(A)$. If $\gamma$ is a weak cycle, i.e., $\gamma = (i)$ for some $i \in N$, then (cf. (2.39)) $B_{\gamma}(A) = \{a_{i,i}\}$. Now, the Brauer Cassini oval $K_{i,j}(A)$, for any $j \neq i$, is, from (2.5),

$$K_{i,j}(A) := \{z \in \mathbb{C} : |z - a_{i,i}| \cdot |z - a_{j,j}| \leq r_{i}(A) \cdot r_{j}(A)\},$$

so that $a_{i,i} \in K_{i,j}(A)$. Thus, $B_{\gamma}(A) \subseteq K_{i,j}(A)$ for any $j \neq i$; whence, $B_{\gamma}(A) \subseteq K(A)$.

Next, assume that $\gamma$ is a strong cycle from $C(A)$, where we point out that the new row sums $\{\tilde{r}_{i}(A)\}_{i=1}^{n}$ for $A$, from (2.37) in the reducible case, and the old row sums $\{r_{i}(A)\}_{i=1}^{n}$ for $A$, necessarily satisfy

$$0 < \tilde{r}_{i}(A) \leq r_{i}(A) \quad \text{(all } i \in \gamma\), \tag{2.49}$$

with $\tilde{r}_{i}(A) = r_{i}(A) > 0$ for all $i \in N$ if $A$ is irreducible. If the strong cycle $\gamma$ has length 2, i.e., $\gamma = (i_{1} \ i_{2})$ where $i_{3} = i_{1}$, then it follows from (2.38) that its associated Brualdi lemniscate is

$$B_{\gamma}(A) = \{z \in \mathbb{C} : |z - a_{i_{1},i_{1}}| \cdot |z - a_{i_{2},i_{2}}| \leq \tilde{r}_{i_{1}}(A) \cdot \tilde{r}_{i_{2}}(A)\}.$$ 

Hence, with (2.5) and the inequalities of (2.49), it follows that

$$B_{\gamma}(A) \subseteq K_{i_{1},i_{2}}(A).$$

Next, assume that this strong cycle $\gamma$ has length $p > 2$, i.e., $\gamma = (i_{1} \ i_{2} \ \cdots \ i_{p})$, with $i_{p+1} = i_{1}$, where the associated new row sums $\{\tilde{r}_{i_{j}}(A)\}_{j=1}^{p}$ are all positive. From (2.38), the associated Brualdi lemniscate is

$$B_{\gamma}(A) := \left\{z \in \mathbb{C} : \prod_{j=1}^{p} |z - a_{i_{j},i_{j}}| \leq \prod_{j=1}^{p} \tilde{r}_{i_{j}}(A) \right\}. \tag{2.50}$$

Let $z$ be any point of $B_{\gamma}(A)$. On squaring the inequality in (2.50), we have
\[ |z - a_{i_1,i_1}|^2 \cdot |z - a_{i_2,i_2}|^2 \cdots |z - a_{i_p,i_p}|^2 \leq \tilde{\rho}_{i_1}^2(A) \cdot \tilde{\rho}_{i_2}^2(A) \cdots \tilde{\rho}_{i_p}^2(A). \]

As these \( \tilde{\rho}_{i_j}(A) \)'s are all positive, we can equivalently express the above inequality as

\[
\frac{|z - a_{i_1,i_1}| \cdot |z - a_{i_2,i_2}|}{\tilde{\rho}_{i_1}(A) \cdot \tilde{\rho}_{i_2}(A)} \cdot \frac{|z - a_{i_3,i_3}|}{\tilde{\rho}_{i_2}(A) \cdot \tilde{\rho}_{i_3}(A)} \cdots \frac{|z - a_{i_p,i_p}|}{\tilde{\rho}_{i_p}(A) \cdot \tilde{\rho}_{i_1}(A)} \leq 1.
\]

As the factors on the left of (2.51) cannot all exceed unity, then at least one of the factors is at most unity. Hence, there is an \( \ell \) with \( 1 \leq \ell \leq p \) such that

\[ |z - a_{i_\ell,i_\ell}| \cdot |z - a_{i_{\ell+1},i_{\ell+1}}| \leq \tilde{\rho}_{i_\ell}(A) \cdot \tilde{\rho}_{i_{\ell+1}}(A), \]

(where if \( \ell = p \), then \( i_{\ell+1} = i_1 \)). But from the definition in (2.5) and from (2.49), we see that \( z \in K_{i_\ell,i_{\ell+1}}(A) \). Hence, as \( z \) is any point of \( B_\gamma(A) \), it follows that

\[ B_\gamma(A) \subseteq \bigcup_{j=1}^{p} K_{i_j,i_{j+1}}(A) \quad (\text{where } i_{p+1} = i_1). \]

Thus, from (2.40) and the above display,

\[ B(A) := \bigcup_{\gamma \in \mathcal{C}(A)} B_\gamma(A) \subseteq \bigcup_{i,j \in \mathbb{N} \atop i \neq j} K_{i,j}(A) := \mathcal{K}(A), \]

the desired result of (2.48).

It is important to remark that while the Brualdi set \( B(A) \) is always a subset (cf.(2.48)) of the Brauer set \( \mathcal{K}(A) \), for any matrix \( A = [a_{i,j}] \in \mathbb{C}^{n \times n} \) with \( n \geq 2 \), there may be substantially more computational work in determining all the Brualdi lemniscate sets \( B_\gamma(A) \), than there is in determining the \( \binom{n}{2} \) Brauer Cassini ovals. That is, getting the possibly sharper inclusion in (2.48) may come at the price of more computations. This is illustrated in the next paragraph.

We next show that there are many cases where equality holds in (2.48) of Theorem 2.9. Consider any matrix \( A = [a_{i,j}] \in \mathbb{C}^{n \times n} \), with \( n \geq 2 \), for which every nondiagonal entry \( a_{i,j} \) of \( A \) is nonzero. The matrix \( A \) is then clearly irreducible. Next, for any \( n \geq 2 \), on defining

\[
\mathcal{P}_n := \{ \text{the set of all cycles of length at least two, from the integers } (1, 2, \cdots, n) \},
\]

it can be verified (see Exercise 2 of this section) that the cardinality of \( \mathcal{P}_n \) (i.e., the number of elements in \( \mathcal{P}_n \)), denoted by \( |\mathcal{P}_n| \), is given by
\[ |\mathcal{P}_n| = \sum_{k=2}^{n} \binom{n}{k} (k - 1)! . \]

Then, each strong cycle \( \gamma \) of \( \mathcal{G}(A) \), given by \( \gamma = (i_1 \ i_2 \ \cdots \ i_p) \) with \( p \geq 2 \), can be associated with an element in \( \mathcal{P}_n \), i.e.,

\[ \gamma = (i_1 \ i_2 \ \cdots \ i_p) \in \mathcal{P}_n, \text{ where } 2 \leq p \leq n. \]

In this case, as each Brauer Cassini oval \( \mathcal{K}_{i,j}(A), i \neq j \), corresponds to a cycle (of length 2) in \( \mathcal{B}(A) \), it follows from (2.40) that \( \mathcal{K}(A) \subseteq \mathcal{B}(A) \), but as the reverse inclusion holds in (2.48), then

\[ \mathcal{B}(A) = \mathcal{K}(A). \]

In other words, the Brualdi set \( \mathcal{B}(A) \) need not, in general, be a proper subset of the Brauer set \( \mathcal{K}(A) \). But what is most striking here is that, for a \( 10 \times 10 \) complex matrix \( A \), all of whose off-diagonal entries are nonzero, there are, from (2.53),

\[ 1,112,073 \text{ distinct cycles in } \mathcal{C}(A). \]

Thus, determining the Brualdi set \( \mathcal{B}(A) \) would require finding 1,112,073 Brualdi lemniscates, a daunting task for a \( 10 \times 10 \) matrix. Fortunately, as \( \mathcal{B}(A) = \mathcal{K}(A) \), only \( \binom{10}{2} = 45 \) of these cycles, corresponding to the Brauer Cassini ovals, are needed to determine \( \mathcal{B}(A) \), while only 10 Geršgorin disks are needed for \( \Gamma(A) \).

The example, in the previous paragraph, where the given irreducible matrix \( A = [a_{i,j}] \in \mathbb{C}^{n \times n}, n > 2 \), has all its off-diagonal entries nonzero, shows that it suffices to consider only the union of the Brauer Cassini ovals for \( A \), in order to obtain a valid eigenvalue inclusion set for \( A \). This reduction is noteworthy, and leads to the following considerations.

Let \( A = [a_{i,j}] \in \mathbb{C}^{4 \times 4} \) be an irreducible matrix with zero diagonal entries, for which \( \gamma_1 = (1 \ 2 \ 3 \ 4) \) and \( \gamma_2 = (1 \ 2 \ 4 \ 3) \) are elements of the cycle set \( \mathcal{C}(A) \). (Note that \( \mathcal{B}_{\gamma_1}(A) = \mathcal{B}_{\gamma_2}(A) \) from (2.38) and Exercise 6 of Section 2.2.) Then, \( A \) is irreducible and its directed graph, based on \( \gamma_1 \) and \( \gamma_2 \), is shown in Fig. 2.5. But, it is evident from this directed graph that \( \gamma_3 = (1 \ 2 \ 3), \gamma_4 = (1 \ 2 \ 4), \text{ and } \gamma_5 = (3 \ 4) \) are the remaining elements of \( \mathcal{C}(A) \). Moreover, using the technique of the proof of Theorem 2.9, it can be verified (see Exercise 3 of this section) that

\[ \mathcal{B}(A) := \bigcup_{j=1}^{5} \mathcal{B}_{\gamma_j}(A) = \mathcal{B}_{\gamma_1}(A) \cup \mathcal{B}_{\gamma_4}(A) \cup \mathcal{B}_{\gamma_5}(A), \]

i.e., the higher-order lemniscates \( \mathcal{B}_{\gamma_1}(A) \) and \( \mathcal{B}_{\gamma_2}(A) \), of order 4, are not needed, from (2.55), in determining \( \mathcal{B}(A) \). From this, we can speak of a reduced cycle set \( \hat{\mathcal{C}}(A) \) of an irreducible matrix \( A \), in which particular higher-order Brualdi lemniscates are omitted from \( \mathcal{C}(A) \), but with \( \hat{\mathcal{C}}(A) \) having the property that...
Fig. 2.5. The directed graph, $\mathbb{G}(A)$, for the cycles $\gamma_1 = (1 \ 2 \ 3 \ 4)$ and $\gamma_2 = (1 \ 2 \ 4 \ 3)$

\[ \bigcup_{\gamma \in \hat{C}(A)} B_\gamma(A) = B(A). \]

We now give the following new result of Brualdi and Varga, Theorem 2.10, which provides a theoretical setting for such cycle reductions. (We note that the reduction of (2.55) is in fact the special case, $m = 2$, of this theorem.)

For notation, if $\gamma = (i_1 \ i_2 \ \cdots \ i_p)$ is a cycle of $C(A)$, then

\[ V(\gamma) := \bigcup_{j=1}^{p} \{i_j\} \text{ is its vertex set}, \]

where we note, in this irreducible case, that all cycles are necessarily strong cycles.

**Theorem 2.10.** Given an irreducible matrix $A = [a_{i,j}] \in \mathbb{C}^{n \times n}, n \geq 2$, let $C(A)$ be its cycle set, and let $\{\gamma_j\}_{j=1}^{s}$, with $s \geq 2$, be distinct cycles of $C(A)$ such that

i) $V(\gamma_1) = \bigcup_{j=2}^{s} V(\gamma_j)$, and

ii) there is a positive integer $m$ such that each vertex from $\gamma_1$ appears exactly $m$ times in $\bigcup_{j=2}^{s} V(\gamma_j)$.

Then (cf.(2.38)),

\[ B_{\gamma_1}(A) \subseteq \bigcup_{j=2}^{s} B_{\gamma_j}(A). \]

**Remark 1.** This means that the Brualdi lemniscate \( B_{\gamma_1}(A) \) can be deleted from the rualdi set (cf.(2.40)) \( B(A) \), since \[ \bigcup_{\gamma \in C(A)/\gamma_1} B_{\gamma}(A) = B(A). \]

**Remark 2.** Noting that the length of a cycle \( \gamma \) in \( C(A) \) is the same as the cardinality of its vertex set \( V(\gamma) \), then the hypotheses of Theorem 2.10 imply that
\[
\text{length}(\gamma_1) \geq \text{length}(\gamma_j), \quad \text{for each } j \text{ with } 2 \leq j \leq s.
\]

Thus, removing \( \gamma_1 \) from \( C(A) \) removes a generally higher-order lemniscate from \( C(A) \).

**Proof.** Since \( A \) is irreducible, then \( r_i(A) > 0 \) for all \( i \in N \). Thus for any \( \gamma \in C(A) \), we can equivalently express its associated Brualdi lemniscate (cf.(2.38)) as
\[ B_{\gamma}(A) = \{ z \in \mathbb{C} : \prod_{i \in \gamma} \left( \frac{|z - a_{i,i}|}{r_i(A)} \right) \leq 1 \}. \]

Now for any \( z \in \mathbb{C} \), the hypotheses i) and ii) of Theorem 2.10 above directly give us that
\[
\prod_{k \in \gamma_1} \left( \frac{|z - a_{k,k}|}{r_k(A)} \right)^m = \prod_{j=2}^{s} \left( \prod_{k \in \gamma_j} \left( \frac{|z - a_{k,k}|}{r_k(A)} \right) \right). \tag{2.57}
\]

Hence, for any \( z \in B_{\gamma_1}(A) \), the product for \( \gamma_1 \), on the left in (2.57), is at most unity. Thus, not all products \( \prod_{k \in \gamma_j} \left( \frac{|z - a_{k,k}|}{r_k(A)} \right) \), for \( 2 \leq j \leq s \) on the right in (2.57), can exceed unity. Therefore, there is an \( i \), with \( 2 \leq i \leq s \), such that \( \prod_{k \in \gamma_i} \left( \frac{|z - a_{k,k}|}{r_k(A)} \right) \leq 1 \), which implies that \( z \in B_{\gamma_i}(A) \), and this gives the desired inclusion of (2.56).

To conclude this section, the use of the Brualdi set \( B(A) \), rather than the Brauer set \( \mathcal{K}(A) \) or the Geršgorin set \( \Gamma(A) \) to estimate the spectrum of \( A \), seems to be more suitable in practical applications in cases where the cycle set \( C(A) \) has few elements, or, more precisely, when its *reduced cycle set* has few elements.

**Exercises**

1. For the tridiagonal \( n \times n \) matrix \( A \), associated with the directed graph of Fig. 1.7, show that its Brualdi set (cf. (2.40)) is, for any \( n \geq 4 \), just
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\[ B(A) = \{ z \in \mathbb{C} : |z - 2| \leq 2 \}. \]

Also, show for \( n \geq 4 \) that \( B(A) = \mathcal{K}(A) = \Gamma(A) \).

2. Prove that the formula in (2.53) is valid. (Hint: Use the result of Exercise 6 of Section 2.2.)

3. For the matrix \( A = [a_{i,j}] \in \mathbb{C}^{4 \times 4} \), whose directed graph is shown in Fig. 2.5, verify that (2.55) is valid. (Hint: Apply Theorem 2.10 for the case \( m = 2 \).)

4. For any complex numbers \( \{ z_j \}_{j=1}^p \) with \( p \geq 2 \), and for any \( p \) nonnegative real numbers \( \{ \rho_j \}_{j=1}^p \), define the Brualdi-like sets

\[ S_p := \left\{ z \in \mathbb{C} : \prod_{j=1}^p |z - z_j| \leq \prod_{j=1}^p \rho_j \right\}, \]

\[ S_p \setminus k := \left\{ z \in \mathbb{C} : \prod_{j=1, j \neq k}^p |z - z_j| \leq \prod_{j=1, j \neq k}^p \rho_j \right\}, \]

for any \( k \) with \( 1 \leq k \leq p \). Then, it is known (Karow (2003)) that

\[ S_p \subseteq \bigcup_{k=1}^p S_p \setminus k. \]

Show that the above inclusion can be deduced as a special case of \( m = p - 1 \) of Theorem 2.10.

2.4 The Sharpness of Brualdi Lemniscate Sets

Given any matrix \( A = [a_{i,j}] \in \mathbb{C}^{n \times n}, n \geq 1 \), we have from (2.42) of Theorem 2.5 that

\[ \sigma(A) \subseteq B(A), \]

where the associated Brualdi set \( B(A) \) is determined, in (2.40), from the quantities

\[ \{a_{i,i}\}_{i=1}^n, \quad \{\tilde{r}_i(A)\}_{i=1}^n, \]

and the cycle set \( \mathcal{C}(A) \) of \( A \).

It is again evident that any matrix \( B = [b_{i,j}] \in \mathbb{C}^{n \times n} \), having the identical quantities of (2.59), has its eigenvalues also in \( B(A) \), i.e., with notations
similar to the equiradial set and the extended equiradial set of (2.15) and (2.16), if

\[(2.60) \quad \omega_B(A) := \{ B = [b_{i,j}] \in \mathbb{C}^{n \times n} : b_{i,i} = a_{i,i}, \tilde{r}_i(B) = \tilde{r}_i(A), \text{ for all } i \in N, \text{ and } C(B) = C(A) \}\]

denotes the **Brualdi radial set** for \( A \), where \( \sigma(\omega_B(A)) := \bigcup_{B \in \omega_B(A)} \sigma(B) \), and if the extended Brualdi radial set is given by

\[(2.61) \quad \hat{\omega}_B(A) := \{ B = [b_{i,j}] \in \mathbb{C}^{n \times n} : b_{i,i} = a_{i,i}, 0 \leq \tilde{r}_i(B) \leq \tilde{r}_i(A), \text{ for all } i \in N, \text{ and } C(B) = C(A) \}\]

for \( A \), where \( \sigma(\hat{\omega}_B(A)) := \bigcup_{B \in \hat{\omega}_B(A)} \sigma(B) \), it follows, in analogy with (2.18), that

\[(2.62) \quad \sigma(\omega_B(A)) \subseteq \sigma(\hat{\omega}_B(A)) \subseteq B(A). \]

(We note that \( \tilde{r}_i(A) > 0 \) for any row \( i \) associated with a strong cycle of \( G(A) \).)

It is of theoretical interest to ask if equality can hold throughout in (2.62). The answer, in general, is no, as the following simple example shows. Consider the matrix

\[(2.63) \quad D = \begin{bmatrix} 1 & 1 & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 0 & 1 \end{bmatrix}, \]

so that

\( r_1(D) = r_2(D) = 1, \) and \( r_3(D) = \frac{1}{2} \).

The directed graph \( G(D) \), without loops, is then given in Fig. 2.6, so that \( D \) is irreducible, and the cycle set of \( D \) is \( C(D) = (1 \ 2) \cup (1 \ 2 \ 3) \). Now, any matrix \( E \) in \( \omega_B(D) \) can be expressed from (2.60) as

\[(2.64) \quad E = \begin{bmatrix} 1 & e^{i\theta_1} & 0 \\ (1-s)e^{i\theta_2} & -1 & se^{i\theta_3} \\ \frac{1}{2}e^{i\theta_4} & 0 & 1 \end{bmatrix}, \]

where \( s \) satisfies \( 0 < s < 1 \) and where \( \{\theta_i\}_{i=1}^4 \) are any real numbers. (Note that letting \( s = 0 \) or \( s = 1 \) in (2.64) does not preserve the cycles of \( C(D) \).) With \( \gamma_1 := (1 \ 2) \) and \( \gamma_2 := (1 \ 2 \ 3) \), we see from (2.38) that
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Fig. 2.6. The directed graph, $\mathcal{G}(D)$, without loops, for the matrix $D$ of (2.63)

\[
\begin{align*}
B_{\gamma_1}(D) &= \{ z \in \mathbb{C} : |z^2 - 1| \leq 1 \}, \text{ and } \\
B_{\gamma_2}(D) &= \{ z \in \mathbb{C} : |z - 1|^2 \cdot |z + 1| \leq 1/2 \},
\end{align*}
\]

where the lemniscate $B_{\gamma_2}(D)$ consists of two disjoint components. These lemniscates are shown in Fig. 2.7.

Fig. 2.7. The lemniscates $B_{\gamma_1}(D)$ and $B_{\gamma_2}(D)$ (shaded) for the matrix $D$ of (2.63)

It can be seen, from (2.65) and from Fig. 2.7, that $z = 0$ is a boundary point of the compact sets $B_{\gamma_1}(D)$ and $B(D) := B_{\gamma_1}(D) \cup B_{\gamma_2}(D)$. Suppose that we can find an $s$ with $0 < s < 1$ and real values of $\{\theta_i\}_{i=1}^4$ for which an associated matrix $E$ of (2.64) has eigenvalue 0. This implies that $\det E = 0$, which, by direct calculations with (2.64), gives

\[
0 = \det E = -1 - (1 - s)^{i(\theta_1 + \theta_2)} + \frac{1}{2}se^{i(\theta_1 + \theta_3 + \theta_4)}, \text{ or}
\]
\[ 1 = \left\{ -(1 - s)e^{i(\theta_1 + \theta_2)} + \frac{1}{2}se^{i(\theta_1 + \theta_3 + \theta_4)} \right\}. \] 

But as \( 0 < s < 1 \), the right side of (2.66) is in modulus at most

\[ (1 - s) + \frac{1}{2}s = \frac{2 - s}{2} < 1, \]

so that \( \det E \neq 0 \) for any \( E \) in \( \omega_B(D) \), i.e., \( 0 \notin \sigma(\omega_B(D)) \). A similar argument shows (cf. (2.61)) that \( 0 \notin \sigma(\hat{\omega}_B(D)) \). But as \( 0 \in B(D) \), we have

\[ \sigma(\omega_B(D)) \subseteq \sigma(\hat{\omega}_B(D)) \subseteq B(D). \] 

But, in order to achieve equality in the last inequality in (2.67), suppose that we allow \( s \) to be zero in (2.64), noting from (2.65), that the parameter \( s \) plays no role in \( B(D) = B_{\gamma_1}(D) \cup B_{\gamma_2}(D) \). Then, on setting \( s = 0 \) in (2.64), the matrix \( E \) of (2.64) becomes

\[ \hat{E} = \begin{bmatrix} 1 & e^{i\theta_1} & 0 \\ e^{i\theta_2} & -1 & 0 \\ \frac{1}{2}e^{i\theta_4} & 0 & 1 \end{bmatrix}, \] 

and on choosing \( \theta_1 = 0 \) and \( \theta_2 = \pi \), then \( z = 0 \) is an eigenvalue of \( \hat{E} \), where \( \hat{E} \) is the limit of matrices \( E \) in (2.64) when \( s \downarrow 0 \). We note that the directed graph of \( G(\hat{E}) \) is shown in Fig. 2.8, so that \( C(\hat{E}) \neq C(D) \), but \( \hat{E} \) remains an element of \( \omega(D) \) of (2.15).

![Fig. 2.8. The directed graph, \( G(\hat{E}) \), with no loops, for the matrix of (2.68)](image)

This example suggests that we consider the closures of the sets \( \omega_B(A) \) and \( \hat{\omega}_B(A) \) of (2.60) and (2.61), where \( A = [a_{i,j}] \in \mathbb{C}^{n \times n}, n \geq 1 \), is any matrix:
\[
\tilde{\omega}_B(A) := \{B = [b_{i,j}] \in \mathbb{C}^{n \times n} : \text{there is a sequence of matrices } \{E_j\}_{j=1}^{\infty} \text{ in } \omega_B(A), \text{ for which } B = \lim_{j \to \infty} E_j\}.
\] (2.69)

and

\[
\hat{\omega}_B(A) := \{B = [b_{i,j}] \in \mathbb{C}^{n \times n} : \text{there is a sequence of matrices } \{E_j\}_{j=1}^{\infty} \text{ in } \hat{\omega}_B(A), \text{ for which } B = \lim_{j \to \infty} E_j\}.
\] (2.70)

This brings us to a recent result (cf. Varga (2001a)) of

**Theorem 2.11.** For any \(A = [a_{i,j}] \in \mathbb{C}^{n \times n}\), then

\[
\partial B(A) \subseteq \sigma(\omega_B(A)) \subseteq \sigma(\hat{\omega}_B(A)) = B(A),
\] (2.71)
i.e., each boundary point of the Brualdi set \(B(A)\) is an eigenvalue of some matrix in \(\omega_B(A)\), and each point of \(B(A)\) is an eigenvalue of some matrix in \(\hat{\omega}_B(A)\).

**Remark:** This establishes the sharpness of the Brualdi set \(B(A)\) for the given matrix \(A\), as the final equality in (2.71) gives that the spectra of matrices in \(\hat{\omega}_B(A)\) are dense in \(B(A)\).

**Proof.** Since \(\sigma(\omega_B(A)) \subseteq \sigma(\hat{\omega}_B(A))\) from (2.62), it follows that their closures, of (2.69) and (2.70), necessarily satisfy \(\sigma(\tilde{\omega}_B(A)) \subseteq \sigma(\hat{\omega}_B(A))\), giving the middle inclusion of (2.71). It suffices to establish the first inclusion and the final equality in (2.71).

First, suppose that \(\gamma\) in \(C(A)\) is a weak cycle. Then, \(\gamma = (i)\) for some \(i \in N\), where its associated Brualdi lemniscate is, from (2.39), \(B_\gamma(A) = \{a_{i,i}\}\). Moreover, from (2.36ii) and (2.35), we see that either \(n = 1\), or \(n \geq 2\) with \(A\) reducible, and that \(a_{i,i}\) is an eigenvalue of \(A\). If all cycles \(\gamma\) in \(C(A)\) are weak cycles, then the Brualdi set \(B(A)\) for \(A\) satisfies \(B(A) = \bigcup_{i=1}^{n} a_{i,i}\). Thus, as this case gives us that \(\tilde{r}_i(A) = 0\) for all \(i \in N\), then with (2.36ii), it follows from (2.34) that each matrix in \(\omega_B(PAP^T)\) or \(\hat{\omega}_B(PAP^T)\) is upper triangular with diagonal entries \(\{a_{i,i}\}_{i=1}^{n}\). Hence,

\[
\partial B(A) = \sigma(\omega_B(A)) = \sigma(\hat{\omega}_B(A)) = B(A) = \bigcup_{i=1}^{n} a_{i,i},
\]

the case of equality in (2.71).

Consider any strong cycle \(\gamma\) in \(C(A)\). From our discussion in Section 2.3, we can express \(\gamma\) as an element of \(P_n\) of (2.52), i.e.,
2.4 The Sharpness of Brualdi Lemniscate Sets

\[(2.72)\]

\[
\gamma = (i_1 \ i_2 \ \cdots \ i_p), \text{ where } 2 \leq p \leq n.
\]

Without loss of generality, we can assume, after a suitable permutation of the rows and columns of \(A\), that

\[(2.73)\]

\[
\gamma = (1 \ 2 \ \cdots \ p),
\]

noting that this permutation leaves unchanged the collection of diagonal entries, row sums, and cycles of \(A\). This permuted matrix, also called \(A\), then has the partitioned form

\[(2.74)\]

\[
A = \begin{bmatrix}
a_{1,1} & \cdots & a_{1,p} & a_{1,p+1} & \cdots & a_{1,n} \\
\vdots & & \vdots & \vdots & & \vdots \\
\frac{a_{p,1}}{a_{p+1,1}} & \frac{a_{p,p}}{a_{p+1,p}} & \frac{a_{p,p+1}}{a_{p+1,p+1}} & \cdots & \frac{a_{p,n}}{a_{p+1,n}} \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
a_{n,1} & \cdots & a_{n,p} & a_{n,p+1} & \cdots & a_{n,n}
\end{bmatrix} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix},
\]

where the matrices \(A_{1,2}, A_{2,1}, \text{ and } A_{2,2}\) are not present in (2.74) if \(p = n\). We also assume, for notational convenience, that \(r_i(A) = \tilde{r}_i(A)\) for \(1 \leq i \leq p\). This means that any entry of \(A_{1,2}\), which arises from an upper triangular block of the normal reduced form of \(A\) in (2.35), is simply set to zero.

Our aim below is to construct a special matrix \(B(t) = [b_{i,j}(t)] \in \mathbb{C}^{n \times n}\), whose entries depend continuously on the parameter \(t\) in \([0, 1]\), such that

\[(2.75)\]

\[
\begin{align*}
b_{i,i}(t) &= a_{i,i}, \quad r_i(B(t)) = r_i(A), \text{ for all } i \in N, \text{ and all } t \in [0, 1], \\
\text{and} \\
C(B(t)) &= C(A) \text{ for all } 0 < t \leq 1.
\end{align*}
\]

To this end, write

\[(2.76)\]

\[
B(t) := \begin{bmatrix}
B_{1,1}(t) & B_{1,2}(t) \\
A_{2,1} & A_{2,2}
\end{bmatrix},
\]

i.e., the rows \(p+1 \leq \ell \leq n\) of \(B(t)\) are exactly those of \(A\), and are independent of \(t\). We note from (2.73) that

\[(2.77)\]

\[
a_{1,2} \cdot a_{2,3} \cdots a_{p-1,p} \cdot a_{p,1} \neq 0,
\]

and the entries of the first \(p\) rows of \(B(t)\) are defined, for all \(t \in [0, 1]\), to satisfy
\[
\begin{align*}
\begin{cases}
\begin{aligned}
b_{i,i}(t) & := a_{i,i} \quad \text{for all } 1 \leq i \leq p; \\
|b_{i,i+1}(t)| & := (1-t)r_i(A) + t|a_{i,i+1}|, \quad \text{and } |b_{i,j}(t)| := t|a_{i,j}| \\
|b_{p,1}(t)| & := (1-t)r_p(A) + t|a_{p,1}|, \quad \text{and } |b_{p,j}(t)| := t|a_{p,j}| \quad (\text{all } j \neq 1, p).
\end{aligned}
\end{cases}
\end{align*}
\tag{2.78}
\]

By definition, the entries of \( B(t) \) are all continuous in the variable \( t \) of \([0, 1]\), and \( B(t) \) and \( A \) have the same diagonal entries. Moreover, it can be verified (see Exercise 1 of this section) that \( B(t) \) and \( A \) have the same row sums for all \( 0 \leq t \leq 1 \), and, as \( a_{i,j} \neq 0 \) implies \( b_{i,j}(t) \neq 0 \) for all \( 0 < t \leq 1 \), then \( B(t) \) and \( A \) have the same cycles in their directed graphs for all \( 0 < t \leq 1 \). Also, from (2.60), \( B(t) \in \omega_B(A) \) for all \( 0 < t \leq 1 \), and from (2.69), \( B(0) \in \omega_B(A) \). Hence, from (2.62),

\[
\tag{2.79}
\sigma(B(t)) \subseteq B(A) \quad \text{for all } 0 < t \leq 1.
\]

But as \( B(A) \) is a closed set from (2.38) - (2.40), and as the eigenvalues of \( B(t) \) are continuous functions of \( t \), for \( 0 \leq t \leq 1 \), we further have, for the limiting case \( t = 0 \), that

\[
\sigma(B(0)) \subseteq B(A),
\]

where, from the definitions in (2.78),

\[
\tag{2.80}
B(0) = \begin{bmatrix}
B_{1,1}(0) & O \\
A_{2,1} & A_{2,2}
\end{bmatrix},
\]

with

\[
\tag{2.81}
B_{1,1}(0) = \begin{bmatrix}
a_{1,1} & r_1(A)e^{i\theta_1} & \cdots & \cdots & \cdots \\
& a_{2,2} & r_2(A)e^{i\theta_2} \\
& & \ddots & \ddots & \ddots \\
& & & a_{p-1,p-1} & r_{p-1}(A)e^{i\theta_{p-1}} \\
r_p(A)e^{i\theta_p} & & & & a_{p,p}
\end{bmatrix}.
\]

We note from (2.78) that the nondiagonal entries in the first \( p \) rows of \( B(t) \) are defined only in terms of their moduli, which allows us to fix the arguments of certain nondiagonal nonzero entries of \( B_{1,1}(0) \) through the factors \( \{e^{i\theta_j}\}_{j=1}^p \) where the \( \{\theta_j\}_{j=1}^p \) are contained in \([0, 2\pi]\). (These factors appear in \( B_{1,1}(0) \) of (2.81).) The partitioned form of \( B(0) \) gives us that

\[
\tag{2.82}
\sigma(B(0)) = \sigma(B_{1,1}(0)) \cup \sigma(A_{2,2}),
\]
and, from the special cyclic-like form of $B_{1,1}(0)$ in (2.81), it is easily seen that each eigenvalue $\lambda$ of $B_{1,1}(0)$ satisfies

$$\prod_{i=1}^{p} |\lambda - a_{i,i}| = \prod_{i=1}^{p} r_i(A),$$

for all real choices of $\{\theta_j\}_{j=1}^{p}$ in $[0, 2\pi]$ in (2.81). But (2.83), when coupled with the definition of $B_{\gamma}(A)$ in (2.38), immediately gives us that $\lambda \in \partial B_{\gamma}(A)$, and, as all different choices of the real numbers $\{\theta_j\}_{j=1}^{p}$, in $B_{1,1}(0)$ of (2.81), give eigenvalues of $B_{1,1}(0)$ which cover the entire boundary of $B_{\gamma}(A)$, we have

$$\bigcup_{\theta_1, \ldots, \theta_p \text{ real}} \sigma(B_{1,1}(0)) = \partial B_{\gamma}(A).$$

This can be used as follows. Let $z$ be any boundary point of $B(A)$ of (2.40). As $B(A)$ is the union of a finite number of closed sets $B_{\gamma}(A)$, this implies that there is a cycle $\gamma$ of $C(A)$ with $z \in \partial B_{\gamma}(A)$, where $B_{\gamma}(A)$ is defined in (2.38). As the result of (2.84) is valid for any $\gamma$ of $C(A)$, then each boundary point $z$ of $B(A)$ is an eigenvalue of some matrix in $\overline{\omega}B(A)$ of (2.69), i.e.,

$$\partial B(A) \subseteq \sigma(\overline{\omega}B(A)),$$

which is the desired first inclusion of (2.71).

To investigate how the eigenvalues of $\overline{\omega}B(A)$ of (2.70) fill out $B(A)$, we modify the definition of the matrix $B(t)$ of (2.75) and (2.78). Let $\{\tau_i\}_{i=1}^{p}$ be any positive numbers such that

$$0 < \tau_i \leq r_i(A) \quad (1 \leq i \leq p),$$

and let $\tilde{B}(t) = [\tilde{b}_{i,j}(t)] \in \mathbb{C}^{n \times n}$ have the same partitioned form as $B(t)$ of (2.76), but with (2.78) replaced by

$$\begin{align*}
\tilde{b}_{i,i}(t) &:= a_{i,i} \text{ for all } 1 \leq i \leq p; \\
|\tilde{b}_{i,j}(t)| &:= \frac{\tau_i}{r_i(A)} |b_{i,j}(t)| \quad (j \neq i), \text{ for } 1 \leq i \leq p, \text{ and } t \in [0, 1].
\end{align*}$$

Then, $\tilde{B}(t)$ and $A$ have the same diagonal entries, the row sums of $\tilde{B}(t)$ now satisfy $r_j(\tilde{B}(t)) = \tau_j$ for all $1 \leq j \leq p$, all $0 \leq t \leq 1$, and $\tilde{B}(t)$ and $A$ have the same cycles for all $0 < t \leq 1$. From (2.61), $\tilde{B}(t) \in \tilde{\omega}B(A)$ for all $0 < t \leq 1$, and from (2.70), $\tilde{B}(0) \in \overline{\omega}B(A)$. In analogy with (2.80), we have

$$\tilde{B}(0) = \begin{bmatrix} \tilde{B}_{1,1}(0) O \\ A_{2,1} O \\ A_{2,2} \end{bmatrix},$$

with
(2.88) \[ \tilde{B}_{1,1}(0) = \begin{bmatrix} a_{1,1} & \tau_1 e^{i\theta_1} \\ a_{2,2} & \tau_2 e^{i\theta_2} \\ \vdots & \ddots \\ \tau_p e^{i\theta_p} & \cdots & a_{p-1,p-1} & \tau_{p-1} e^{i\theta_{p-1}} \\ \end{bmatrix}, \]

where

\[ \sigma(\tilde{B}(0)) = \sigma(\tilde{B}_{1,1}(0)) \cup \sigma(A_{2,2}). \tag{2.89} \]

It similarly follows that any eigenvalue \( \lambda \) of \( \tilde{B}_{1,1}(0) \) in (2.88) now satisfies

\[ \prod_{j=1}^{p} |\lambda - a_{i,j}| = \prod_{i=1}^{p} \tau_i, \tag{2.90} \]

for any choice of the real numbers \( \{\theta_j\}_{j=1}^p \) in \( \tilde{B}_{1,1}(0) \) of (2.88). Writing

\[ \tilde{B}_{1,1}(0) = \tilde{B}_{1,1}(0; \tau_1, \cdots, \tau_p; \theta_1, \cdots, \theta_p) \]

to show this matrix’s dependence on these parameters \( \tau_i \) and \( \theta_i \), we use the fact that \( \{\tau_i\}_{i=1}^p \) are any numbers satisfying (2.86) and that \( \{\theta_i\}_{i=1}^p \) are any real numbers in \([0, 2\pi]\). Hence, it follows, from the definition of \( B_\gamma(A) \) in (2.50) and closure considerations, that all the eigenvalues of \( \tilde{B}_{1,1}(0; \tau_1, \cdots, \tau_p; \theta_1, \cdots, \theta_p) \) fill out \( B_\gamma(A) \), i.e.,

\[ \left\{ \bigcup_{\{0<\tau_i \leq \tau_i(A)\}_{i=1}^p \{\theta_i\}_{i=1}^p \in [2\pi]} \sigma(\tilde{B}_{1,1}(0; \tau_1, \cdots, \tau_p; \theta_1, \cdots, \theta_p)) \right\} = B_\gamma(A). \tag{2.91} \]

As this holds for any \( \gamma \in C(A) \), where \( \tilde{B}(0) \in \tilde{\omega}_B(A) \), then

\[ \sigma(\tilde{\omega}_B(A)) = B(A), \tag{2.92} \]

the desired final result of (2.71).

Exercises

1. Verify that \( B(t) \), of (2.78), and \( A \) of (2.76) have the same row sums for all \( 0 \leq t \leq 1 \), and the same cycles in their directed graphs for all \( 0 < t \leq 1 \).

2. Show, for all real choices of \( \{\theta_j\}_{j=1}^p \) in \([0, 2\pi]\) in (2.81), that the eigenvalues of \( B_{1,1}(0) \) in (2.81) fill out \( \partial B_\gamma(A) \) in (2.84).
3. Show, for all real choices of \( \{\theta_j\}_{j=1}^\rho \) in \([0, 2\pi]\) and all choices of \( \{\tau_j\}_{j=1}^\rho \) satisfying (2.86), that the eigenvalues of \( \tilde{B}_{1,1}(0) \) of (2.88) are dense in \( \mathcal{B}_\gamma(A) \) of (2.38).

4. Consider the familiar \( n \times n \) irreducible tridiagonal matrix

\[
A = \begin{bmatrix}
2 & -1 & & \\
-1 & 2 & -1 & \\
& \ddots & \ddots & \ddots \\
& & -1 & 2 & -1 \\
& & & -1 & 2
\end{bmatrix},
\]

whose Brualdi set, from Exercise 1 of Section 2.3, is \( \mathcal{B}(A) = \{z \in \mathbb{C} : |z - 2| \leq 2\} \) for any \( n \geq 4 \). While \( z = 0 \) is a boundary point of \( \mathcal{B}(A) \), show, using Theorem 2.8, that \( z = 0 \) is not an eigenvalue of \( A \). However, show that \( z = 0 \) is an eigenvalue of a specific matrix in \( \mathcal{W}_B(A) \) of (2.69), for any \( n \geq 4 \).

5. Assume that \( A = [a_{i,j}] \in \mathbb{C}^{n \times n}, n \geq 2, \) is irreducible and that its cycle set \( \mathcal{C}(A) \) consists of only one cycle, \( \gamma = (1 \ 2 \ \cdots \ n) \). Show in this case (cf. (2.71)) that

\[
\partial \mathcal{B}(A) \neq \sigma(\omega_B(A)) \subset \mathcal{B}(A).
\]

2.5 An Example

To illustrate the result of Theorem 2.11, consider the matrix

\[
E = \begin{bmatrix}
1 & e^{i\theta_1} & 0 & 0 \\
e^{i\theta_2} & i & e^{i\theta_3} & 0 \\
0 & 0 & -1 & e^{i\theta_4} \\
e^{i\theta_5} & 0 & 0 & -i
\end{bmatrix},
\]

where \( \{\theta_i\}_{i=1}^5 \) are any real numbers in \([0, 2\pi]\). Then, \( E \) is irreducible, with cycle set \( \mathcal{C}(E) = (1 \ 2) \cup (1 \ 2 \ 3 \ 4) \), and with row sums \( r_i(E) = 1 \) for all \( 1 \leq i \leq 4 \). In this case, we have from (2.40) that the Brualdi set \( \mathcal{B}(E) \) is the union of the two closed lemniscates.
Fig. 2.9. The lemniscates, $B_{\gamma_1}(E)$ and $B_{\gamma_2}(E)$, for the matrix $E$ of (2.93)

$$
\begin{align*}
B_{\gamma_1}(E) := \{z \in \mathbb{C} : |z - 1| \cdot |z - i| \leq 1\} &= K_{1,2}(E), \\
\text{and} \\
B_{\gamma_2}(E) := \{z \in \mathbb{C} : |z^4 - 1| \leq 1\}.
\end{align*}
$$

These sets are shown in Fig. 2.9, where $B_{\gamma_2}(E)$ has the shape of a four-leaf clover.

To show how the eigenvalues of $E$ fill out $B(E)$, we take random numbers $s$ from the open interval $(0, 1)$ and random values of $\{\theta_i\}_{i=1}^5$ from $[0, 2\pi]$, and, using Matlab 6, the eigenvalues of these matrices are plotted in Fig. 2.10. Fig. 2.10 shows indeed that these eigenvalues of $A$ tend to fill-out $B(E)$.

The following near paradox arose from Theorem 2.11. As an example, the matrix $E$ of (2.93) is irreducible, and it is known from Theorem 2.8 that a necessary condition for a boundary point $z$ of $B(E)$ to be an eigenvalue of $E$ is that $z$ is a boundary point of each of the lemniscates $B_{\gamma_1}(E)$ and $B_{\gamma_2}(E)$ of (2.94). (This is a generalization of Taussky’s Theorem 1.12 on Gershgorin disks, to lemniscates.) But from Fig. 2.9, it is apparent that $z = 0$ is the only point for which $\partial B_{\gamma_1}(E)$ and $\partial B_{\gamma_2}(E)$ have a common point. Yet, (2.71) of Theorem 2.11 gives the nearly contradictory result that each point of $\partial B(E)$ is an eigenvalue of some matrix in $\mathcal{G}(E)$. The difference, of course, lies in the fact that the result of Theorem 2.8 applies to the fixed matrix.
$E$, while the common data of (2.59) applies to all matrices which lie in the closure of $\omega_B(E)$. 
Bibliography and Discussion

2.1. Ostrowski (1937b) first obtained the nonsingularity result of Theorem 2.1. Brauer (1947) later independently obtained the equivalent eigenvalue inclusion result of Theorem 2.2 by means of a direct proof. (Brauer, in the same paper, obtained Ostrowski’s Theorem 2.1 by constructively deriving, from the hypothesis of (2.1), that \( \min \{ |\lambda| : \lambda \in \sigma(A) \} > 0 \).) Though well-known today, the equivalence of these two results was not widely recognized until several years after Brauer’s paper in 1947.

Brauer’s name had become synonymous with the ovals of Cassini, but, for unknown reasons, they are rarely mentioned, even though they are superior (cf. Theorem 2.3) to the Geršgorin disks. (An exception is the book by Korganoff (1961).) Even today, they have not been widely utilized, most likely because i) there are \( n(n-1)/2 \) such ovals, as compared with \( n \) Geršgorin disks, and ii) these ovals are, by their definition, more complicated than disks. The author, however, hopes that this book will generate more interest and applications for this area.

For a further corroboration of Theorem 2.3, as in Fig. 2.2, where the Geršgorin set \( \Gamma(A) \) with the Brauer set \( \mathcal{K}(A) \) are compared, we recommend the following interactive supplement. For an arbitrary \( 3 \times 3 \) complex matrix, go to the website:

http://etna.mcs.kent.edu

for the electronic journal ETNA (Electronic Transactions on Numerical Analysis), click on volume 8 (1999), and go to the paper by Varga and Krautstengl (1999), “On Geršgorin-type problems and ovals of Cassini”, 15-20. Then click on the Interactive Supplement, which was written in Java by Dr. Bryan Lewis. There, for a \( 3 \times 3 \) complex matrix of your choice, or a \( 3 \times 3 \) complex matrix which is generated randomly, one gets the associated spectrum of this matrix, Geršgorin disks (in color), and the Cassini ovals (also in color)!

The fact the Brauer’s ovals of Cassini do a perfect job in estimating the spectra of all matrices \( \omega(A) \) of (2.15) for any matrix \( A = [a_{i,j}] \in \mathbb{C}^{n \times n}, n \geq 3 \), was shown recently by Varga and Krautstengl (1999), though the exact result of (2.19) of Theorem 2.4 appeared earlier in Theorem 6.15 of Engel (1973), with essentially the same proof. This paper of Engel (1973) is a very important contribution, as it derives results for column and row linear matrix functions, which includes determinants and permanents.
2.2 Part of the disappointment in using ovals of Cassini came possibly from the natural urge to march forward, beyond ovals of Cassini, to higher order lemniscates with the hope of superior eigenvalue inclusion results. This, unfortunately, failed in simple cases, as in the case of the $4 \times 4$ matrix in (2.33). Subsequently, Brualdi, in 1982, brilliantly showed how to “solve” this problem by using cycles from the directed graph of a matrix. In essence, Brualdi (1982) derived his related result of Theorem 2.5 for weakly irreducible matrices, which are matrices having only strong cycles (see Exercise 1 of Section 2.2). The Brualdi sets of (2.40), which are a new but modest generalization of Brualdi’s work, include the notion of weak cycles, which permits all matrices to be analyzed from a knowledge of their directed graphs. (Our proof, given in Theorem 2.5, is perhaps simpler than Brualdi’s, as it avoids Brualdi’s use of partial orderings.

We also mention that Karow (2003), in his Ph.D. thesis, also effectively developed the analog of weak cycles in his definition of $\sigma_0(A)$, used in his Theorem 4.6.4, which is also an extension of Brualdi’s work. This extension is done without using the normal reduced form of (2.35), which, however, is needed for the sharpness in Theorem 2.11 of this chapter.

As in Chapter 1, we remark that (2.45) of Theorem 2.8 gives, for an irreducible matrix $A$ in $\mathbb{C}^{n \times n}$, a necessary condition for an eigenvalue $\lambda$ of $A$ to lie on the boundary of the Brualdi set $B(A)$. In this regard, important related necessary and sufficient for this to happen, for a given fixed matrix, have been given in Li and Tsatsomeros (1997) and Kolotololina (2001), (2003a) and (2003b). Our interest here, as is apparent, has been in the related, but diametrically opposed, problem of seeing if each point of an eigenvalue inclusion set is an eigenvalue of some matrix associated with that eigenvalue inclusion set.

2.3 The comparison of the Brualdi sets with the Brauer sets was carried out recently in Theorem 2.6 in Varga (2001b).

The notion of a reduced cycle set, in this section, is new. The result of Theorem 2.10, which gives sufficient conditions for replacing a cycle of an irreducible matrix by lower-order cycles, is an unpublished consequence of an exciting exchange of e-mails with Richard Brualdi, for which the author is most thankful.
It is interesting to mention that Brauer (1952) gave in his Theorem 22 an **erroneous** result, patterned after Taussky’s Theorem 1.12, which stated that if \( A = [a_{i,j}] \in \mathbb{C}^{n \times n}, \ n \geq 2, \) is irreducible, then \( \lambda, \) a boundary point of the union of its associated Cassini ovals of (2.6), can be an eigenvalue of \( A \) only if \( \lambda \) is a boundary point of each of the \( n(n-1)/2 \) ovals of Cassini \( K_{i,j}(A) \) of (2.5). This error was undetected, in the current literature of widely available journals, until Zhang and Gu (1994) gave a simple \( 3 \times 3 \) matrix counterexample, which is given in Exercise 6 of Section 2.2. But, it was kindly pointed out to me recently by Ludwig Elsner that such a counterexample was earlier published by his student Rein (1967) in the more obscure Kleine Mitteilungen of the journal Zeit. Angew. Math. Mech. What is also interesting is that the counterexample of Zhang and Gu is a \( 3 \times 3 \) matrix whose first and third rows are identical to the earlier \( 3 \times 3 \) matrix of Rein, while the second row of the Zhang/Gu matrix is exactly a multiple of 2 of the corresponding Rein matrix!

It should also be quietly mentioned that Feingold and Varga (1962) used Brauer’s incorrect result to obtain in a “generalization” to partitioned matrices, which is also incorrect, but now easily corrected in Chapter 6.

2.4 The sharpness of the Brualdi lemniscate sets, as given in Theorem 2.11, comes from Varga (2001a).