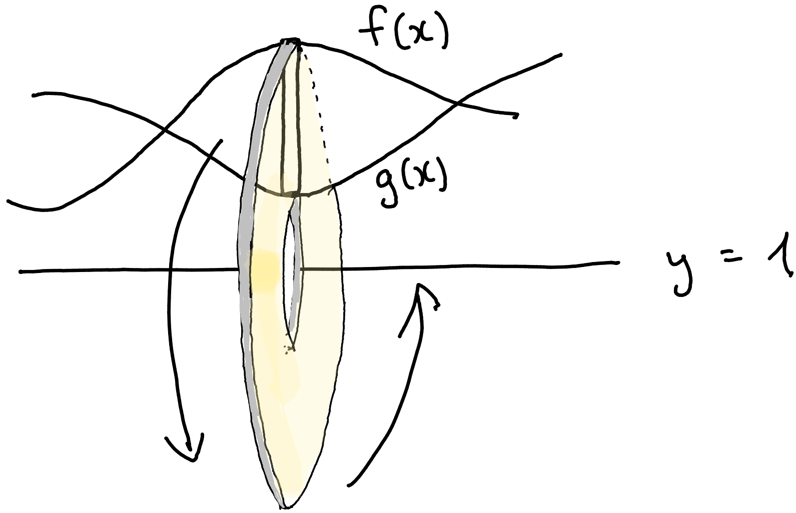
Washer and Shell Method

1 Washer Method

Suppose we want to rotate the region shown below around the given line. To use the washer method, we chop the region up into very narrow rectangles whose longer side perpendicular to the given line. When a single rectangle is rotated around the line, it creates a "washer" or "disk". Together, all these disks fill up the volume created by the rotation.



The volume of one of these disks can be found by subtracting the volume of the smaller disk from that of the larger disk! The volume of a single disk is given by $\pi r^2 h$ where r is the radius of rotation and h is the thickness. If the thickness is denoted dx, then:

- The volume of the outer disk is $\pi(f(x) - l)^2 dx$
- The volume of the inner disk is $\pi(g(x) - l)^2 dx$

so subtracting gives us

$$\pi((f(x) - l)^2 - (g(x) - l)^2)dx$$

To sum up the volumes of all of these disks, we use integration:

$$\text{Volume} = \pi \int_a^b ((f(x) - l)^2 - (g(x) - l)^2) dx$$

The washer method can be applied in a number of situations – for this reason, the formula you remember for the washer method should be quite general.

The key points are the following:

- Use the washer method when splitting the region into **rectangles that are perpendicular to the line of rotation.**
- Remember the formula as

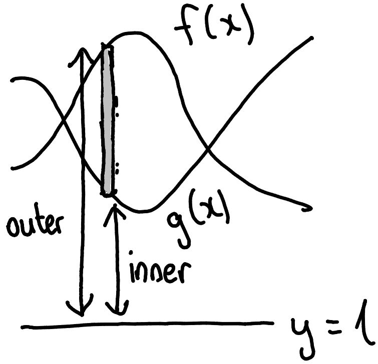
$$\boxed{\text{Volume} = \pi \int_a^b ((\text{outer radius})^2 - (\text{inner radius})^2) dx}$$

The outer radius and inner radius are found in different ways depending on the position of the line of rotation with respect to the area being rotated. For example, if we are dealing with functions of x, we have two cases:

Case 1:

Outer radius: $f(x) - l$

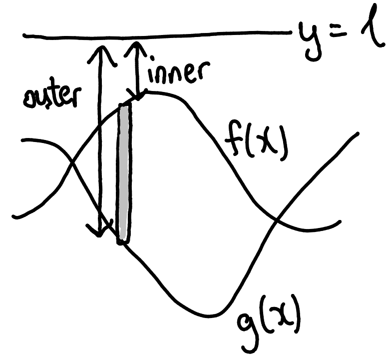
Inner radius: $g(x) - l$



Case 2:

Outer radius: $l - g(x)$

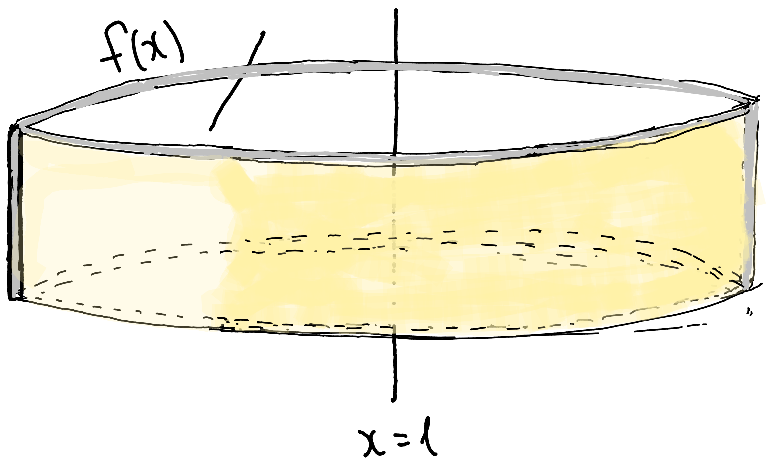
Inner radius: $l - f(x)$



Exercise: Try to figure out the outer radius and inner radius for the two cases when using the washer method when working with functions of y.

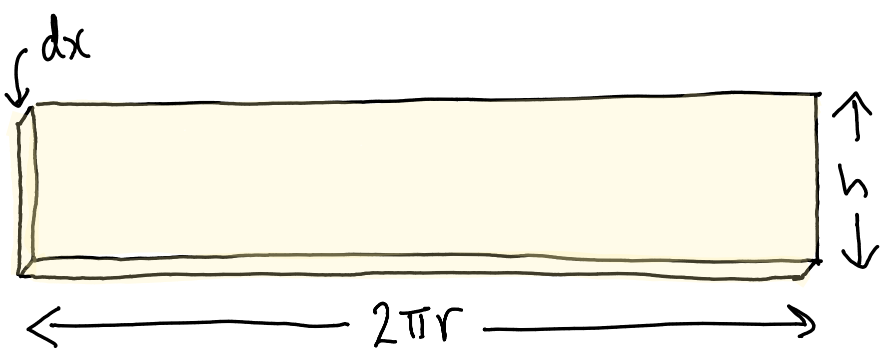
2 Shell Method

The shell method is another method of calculating a volume obtained from rotating an area around a line. **The shell method is used when the region to be rotated is chopped into rectangles whose longer sides are parallel to the line of rotation.**
It is called the shell method, because rotation of a rectangle around a line parallel created a shell this time, not a disk:



To use the shell method, we first must find out how to calculate the volume of one shell. We can then use integration to sum the volumes of all shells.

Imagine the shell above cut and flattened out as shown in the diagram below.



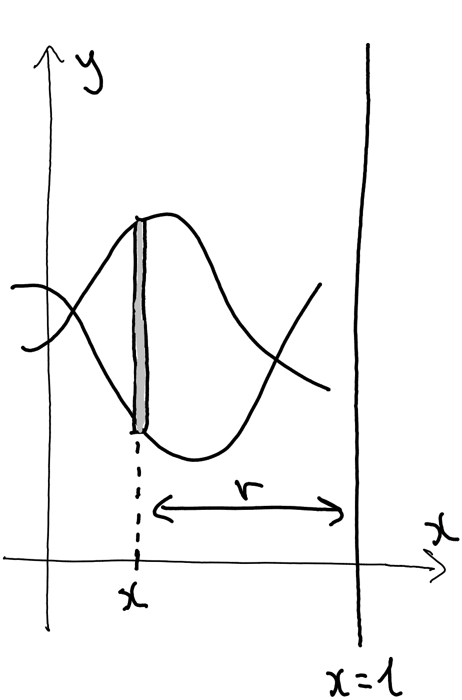
The volume of the above shape is given by the formula $(2\pi r)(h)(dx)$, since the width of the rectangle corresponds to the circumference of the shell, which is $2\pi r$ the height is h and the width is described by dx. Hence, if this is the volume of one shell, summing over all the shells, we get

$$\boxed{\text{Volume } = 2\pi \int_a^b (r)(h)dx}$$

When using the shell method, the height of the shells h will always be found by subtracting the lower function from the upper function. However, the radius of rotation will be different depending on which side of the area the line is located. See the cases below:

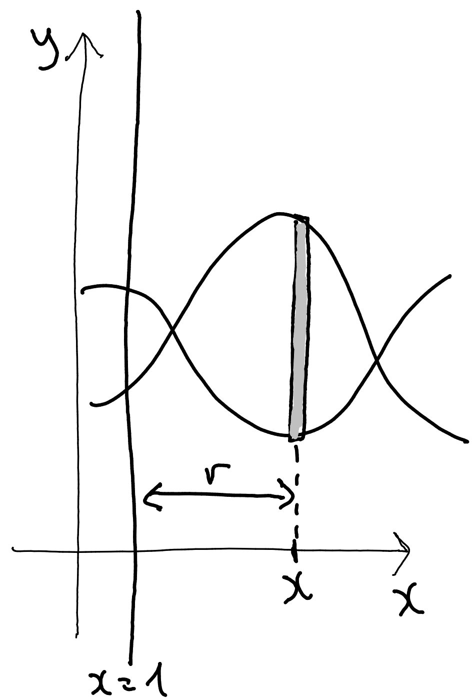
<u>Case 1:</u>

Radius: $l - x$



<u>Case 2:</u>

Radius: $x - l$



<u>Exercise:</u> Try to figure out the radius for the two cases when using the shell method when working with functions of y.

Exercises

1. Using the appropriate method, write down an integral describing the volume obtained when the enclosed is rotated around each of the lines below.

   (i) $x = 10$

   (ii) $x = -8$

   (iii) $y = 15$

   (iv) $y = -9$

   For each of the following regions, practise changing them to functions of y and performing the calculation again.

   (a) $y = \sqrt{x}$ and $y = x^3$.

   (b) $y = 2^x$, $y = 8 - x^2$, the x-axis and the y-axis.
   Hint: it may be difficult to solve the intersection of 2^x and $8 - x^2$ algebraically, but you should be able to identify it by inspection. It might help to graph the functions.

Come to the MHC for help if you are stuck on any of these problems!