Math Help Centre

Trigonometric Substitutions

A trigonometric substitution is required when trying to integrate a functions with a terms like

$$\sqrt{a^2-b^2x^2}, \quad \sqrt{a^2+b^2x^2} \quad \text{or} \quad \sqrt{b^2x^2-a^2}.$$

These terms are tackled using the identities

$$\begin{aligned} 1-\sin^2 x &= \cos^2 x \\ 1+\tan^2 x &= \sec^2 x \\ \sec^2 x - 1 &= \tan^2 x \end{aligned}$$

If the expression inside the square root can be transformed into, for example, something resembling $1+\tan^2 x$, we can use the second identity to replace it with $\sec^2 x$ which helps us simplify the integral.

Here's how the substitutions work:

Case 1:

For $\sqrt{a^2-b^2x^2}$, let $x=\frac{a}{b}\sin\theta$. Then we have,

$$\begin{aligned} &\sqrt{a^2-b^2x^2} \\ =&\sqrt{a^2-b^2\left(\frac{a}{b}\sin\theta\right)^2} \\ =&\sqrt{a^2-b^2\left(\frac{a^2}{b^2}\sin^2\theta\right)} \\ =&\sqrt{a^2-a^2\sin^2\theta} \\ =&\sqrt{a^2(1-\sin^2\theta)} \\ =&a\sqrt{\cos^2\theta} \\ =&a|\cos\theta| \end{aligned}$$

For this substitution, $\theta=\sin^{-1}\left(\frac{bx}{a}\right)$, so $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. Therefore, $a|\cos\theta|=a\cos\theta$

Case 2:

For $\sqrt{a^2+b^2x^2}$, let $x=\frac{a}{b}\tan\theta$. Then we have,

$$\begin{aligned} &\sqrt{a^2+b^2x^2} \\ =&\sqrt{a^2+b^2\left(\frac{a}{b}\tan\theta\right)^2} \\ =&\sqrt{a^2+b^2\left(\frac{a^2}{b^2}\tan^2\theta\right)} \\ =&\sqrt{a^2+a^2\tan^2\theta} \\ =&\sqrt{a^2(1+\tan^2\theta)} \\ =&a\sqrt{\sec^2\theta} \\ =&a|\sec\theta| \end{aligned}$$

For this substitution, $\theta=\tan^{-1}\left(\frac{bx}{a}\right)$, so $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. Therefore, $a|\sec\theta|=a\sec\theta$

Case 3:

For $\sqrt{b^2x^2-a^2}$, let $x=\frac{a}{b}\sec\theta$. Then we have,

$$\begin{aligned} &\sqrt{b^2x^2-a^2} \\ =&\sqrt{b^2\left(\frac{a}{b}\sec\theta\right)^2-a^2} \\ =&\sqrt{b^2\left(\frac{a^2}{b^2}\sec^2\theta\right)-a^2} \\ =&\sqrt{a^2\sec^2\theta-a^2} \\ =&\sqrt{a^2\left(\sec^2\theta-1\right)} \\ =&a\sqrt{\tan^2\theta} \\ =&a|\tan\theta| \end{aligned}$$

For this substitution, $\theta=\sec^{-1}\left(\frac{bx}{a}\right)$, so $0\le\theta<\frac{\pi}{2},\ \pi\le\theta<\frac{3\pi}{2}$. Therefore, $a|\tan\theta|=a\tan\theta$

Example. *Solve the integral* $\int \frac{1}{x\sqrt{x^2+3}}dx$.

<u>*Solution:*</u>

$$\int \frac{1}{x\sqrt{x^2+3}}dx = \int \frac{1}{x\sqrt{\left(\sqrt{3}\right)^2+(1)^2x^2}}dx$$

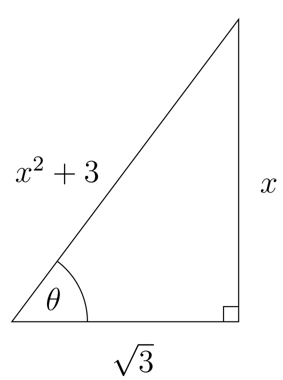
so we can see that we have a term in the interval of the form $\sqrt{a^2+b^2x^2}$. Hence, we use the substitution described in Case 2 above.

Let $x = \frac{\sqrt{3}}{1}\tan\theta = \sqrt{3}\tan\theta$. Then, $dx = \sqrt{3}\sec^2\theta d\theta$, and our integral becomes

$$\begin{aligned}
\int \frac{1}{x\sqrt{x^2+3}}dx &= \int \frac{1}{\sqrt{3}\tan\theta\sqrt{3\tan^2\theta+3}}\sqrt{3}\sec^2\theta d\theta \\
&= \int \frac{1}{\cancel{\sqrt{3}}\tan\theta\sqrt{3(\tan^2\theta+1)}}\cancel{\sqrt{3}}\sec^2\theta d\theta \\
&= \int \frac{\sec^2\theta}{\tan\theta\sqrt{3\sec^2\theta}}d\theta \\
&= \frac{1}{\sqrt{3}}\int \frac{\sec^{\cancel{2}}\theta}{\tan\theta\cancel{\sec\theta}}d\theta \\
&= \frac{1}{\sqrt{3}}\int \frac{\sec\theta}{\tan\theta}d\theta \\
&= \frac{1}{\sqrt{3}}\int \csc\theta d\theta \\
&= \frac{1}{\sqrt{3}}\ln|\csc\theta-\cot\theta|+C
\end{aligned}$$

But we are not yet finished. Our integral started off with x as the variable, so we want to return our answer in terms of x.

From before, we had $x = \frac{\sqrt{3}}{1}\tan\theta \implies \tan\theta = \frac{x}{\sqrt{3}}$. Using a right-angled triangle (which we complete using the fact that $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$ and applying Pythagorus' Theorem to find the hypotenuse), we can figure out expressions for $\csc\theta$ and $\cot\theta$ in terms of x as well.



$\csc\theta = \dfrac{x^2+3}{x}$ and $\cot\theta = \dfrac{\sqrt{3}}{x}$, so our final answer can becomes

$$\frac{1}{\sqrt{3}}\ln|\csc\theta - \cot\theta| + C = \frac{1}{\sqrt{3}}\ln\left|\frac{x^2+3}{x} - \frac{\sqrt{3}}{x}\right| + C$$

Example. *Solve the integral* $\displaystyle\int \frac{\sqrt{9-x^2}}{x^4}dx$.

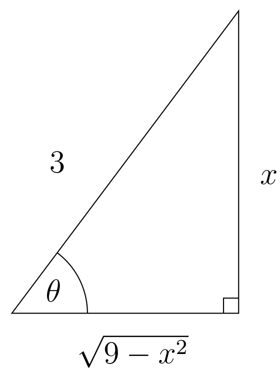
Solution:
This substitution required for this question is the one described in Case 1.
Let $x = 3\sin\theta$. Then we get $dx = 3\cos\theta d\theta$.

$$\begin{aligned}
\int \frac{\sqrt{9-x^2}}{x^4}dx &= \int \frac{\sqrt{9-(3\sin\theta)^2}}{(3\sin\theta)^4}3\cos\theta d\theta \\
&= \int \frac{\sqrt{9-9\sin^2\theta}}{3^4\sin^4\theta}3\cos\theta d\theta \\
&= \int \frac{3\sqrt{\cos^2\theta}}{3^4\sin^4\theta}3\cos\theta d\theta \\
&= \int \frac{\cancel{3^2}\cos^2\theta}{3^{\cancel{4}}\sin^4\theta}d\theta \\
&= \frac{1}{3^2}\int \frac{\cos^2\theta}{\sin^4\theta}d\theta \\
&= \frac{1}{9}\int \frac{\cos^2\theta}{\sin^2\theta}\cdot\frac{1}{\sin^2\theta}d\theta \\
&= \frac{1}{9}\int \cot^2\theta\cdot\csc^2\theta d\theta \\
&= -\frac{1}{9}\int \cot^2\theta\cdot -\csc^2\theta d\theta
\end{aligned}$$

Letting $u = \cot\theta$, we get $du = \csc^2\theta d\theta$, so

$$\begin{aligned}
-\frac{1}{9}\int \cot^2\theta\cdot -\csc^2\theta d\theta &= -\frac{1}{9}\int u^2 du \\
&= -\frac{1}{9}\cdot\frac{u^3}{3} + C \\
&= -\frac{1}{9}\cdot\frac{\cot^3\theta}{3} + C
\end{aligned}$$

Again, we want to give our answer in terms of x, not u or θ, so we will use the right-angled triangle as before. We had $x = 3\sin\theta$, so $\sin\theta = \dfrac{x}{3}$ and the triangle becomes



From the triangle, $\tan\theta = \dfrac{x}{\sqrt{9-x^2}}$, so $\cot\theta = \dfrac{\sqrt{9-x^2}}{x}$, and our final answer becomes

$$-\frac{1}{9}\cdot\frac{\cot^3\theta}{3} + C = -\frac{\sqrt{9-x^2}}{27x} + C$$

Example. *Solve the integral* $\displaystyle\int \frac{2}{x^2-4x}dx.$

Solution:

Although this integral can be solved using partial fractions, a trig substitution can also be solved. In order for a trig substitution to be used, we need something in the integrand to be in one of the following forms:

$$a^2 - b^2x^2, \qquad a^2 + b^2x^2, \qquad \text{or} \qquad b^2x^2 - a^2$$

We do this by completing the square:

$$\begin{aligned} x^2 - 4x &= x^2 - 4x + 4 - 4 \\ &= (x-2)^2 - 4 \end{aligned}$$

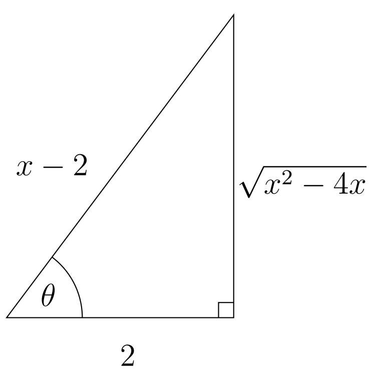
Letting $u = x - 2$, our integral becomes

$$\int \frac{2}{x^2-4x}dx = \int \frac{2}{u^2-4}du$$

Now, letting $2\sec\theta = u$, we get $du = 2\sec\theta\tan\theta$, so

$$\begin{aligned}
\int \frac{2}{u^2-4}du &= \int \frac{2}{(2\sec\theta)^2-4} \cdot 2\sec\theta\tan\theta d\theta \\
&= \int \frac{4\sec\theta\tan\theta}{4\sec^2\theta - 4}d\theta \\
&= \int \frac{\cancel{4}\sec\theta\tan\theta}{\cancel{4}(\sec^2\theta - 1)}d\theta \\
&= \int \frac{\sec\theta\cancel{\tan\theta}}{\tan^{\cancel{2}}\theta}d\theta \\
&= \int \frac{\sec\theta}{\tan\theta}d\theta \\
&= \int \sec\theta \cdot \frac{1}{\tan\theta}d\theta \\
&= \int \frac{1}{\cancel{\cos\theta}} \cdot \frac{\cancel{\cos\theta}}{\sin\theta}d\theta \\
&= \int \csc\theta d\theta \\
&= \ln|\csc\theta - \cot\theta| + C
\end{aligned}$$

But we had $\sec\theta = \frac{u}{2} = \frac{x-2}{2}$, so using a right-angled triangle,



From the triangle, $\csc\theta = \frac{x-2}{\sqrt{x^2-4x}}$ and $\cot\theta = \frac{2}{\sqrt{x^2-4x}}$, so our final answer becomes

$$\ln\left|\frac{x-2}{\sqrt{x^2-4x}} - \frac{2}{\sqrt{x^2-4x}}\right| + C$$

Exercises

Calculate each of the following integrals

1. $\displaystyle\int \frac{1}{\sqrt{9-x^2}}dx$

2. $\displaystyle\int \frac{3}{\sqrt{5-3x^2}}dx$

3. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-25}}dx$

4. $\displaystyle\int x^3(5x^2-2)^{\frac{5}{2}}dx$

5. $\displaystyle\int \frac{\sqrt{x^2+16}}{x^4}dx$

6. $\displaystyle\int 2x^5\sqrt{7+3x^2}dx$

Come to the MHC for help with any of these problems!