Math Help Centre

Trigonometric Integrals

When trying to solve a trigonometric integral, it is helpful to think of the following trig functions as paired:

- $\sin x$ and $\cos x$
- $\tan x$ and $\sec x$
- $\cot x$ and $\csc x$

There are two reasons for this:

1. Each appears in the other's derivative.
2. Each are linked by a Pythagorean identity. Recall:
   - $\sin^2 x + \cos^2 x = 1$
   - $\tan^2 x + 1 = \sec^2 x$
   - $1 + \cot^2 x = \csc^2 x$

These are the keys to solving trig integrals. When given a trigonometric integral, it may be necessary to use trig identities in order to have your integral only contain the above paired trig functions.

Let's start with integrals containing $\sin x$ and $\cos x$. Consider the integral:

$$\int \sin^m x \cos^n x dx$$

1. If m is odd, rearrange the integral to contain $\sin x dx$ at the end, and use the identity $(\sin^2 x)^k = (1 - \cos^2 x)^k$. Now the integral is in the form $\int f(\cos x) \cdot \sin x dx$, so the substitution $u = \cos x$ will solve the integral.
2. If n is odd, rearrange the integral to contain $\cos x dx$ at the end, and use the identity $(\cos^2 x)^k = (1 - \sin^2 x)^k$, and the substitution $u = \sin x$ will work.
3. If both n and m are odd, then either of the above suggestions will work.
4. If both m and n are even, use the identities

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \qquad \text{and/or} \qquad \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

These identities may need to be used repeatedly.

Example. *Solve the integral*

$$\int \sin^3 x \cos^4 x dx$$

Solution:
This integral falls into the first category – m is odd and n is even, so let's rearrange the integral as we were told and use the trig identity $(\sin^2 x)^k = (1-\cos^2 x)^k$ (in this case, $k=1$):

$$\begin{aligned}\int \sin^3 x \cos^4 x dx &= \int \sin^2 x \cos^4 x \cdot \sin x dx \\ &= \int (1-\cos^2 x) \cos^4 x \cdot \sin x dx \\ &= \int (\cos^4 x - \cos^6 x) \cdot \sin x dx\end{aligned}$$

Letting $u = \cos x$, we get $du = -\sin x dx$, so our integral becomes

$$\begin{aligned}\int (\cos^4 x - \cos^6 x) \cdot \sin x dx &= -\int u^4 - u^6 du \\ &= -\frac{u^5}{5} + \frac{u^7}{7} + c \\ &= -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + c\end{aligned}$$

Example. *Solve the integral*

$$\int \cos^4 x dx$$

Solution:
This question falls into the fourth category above – both m and n are even ($m = 0$ and 0 is an even number).

$$\begin{aligned}\int \cos^4 x dx &= \int (\cos^2 x)^2 dx \\ &= \int \frac{1}{2}(1+\cos 2x)^2 dx \\ &= \frac{1}{2}\int 1 + 2\cos 2x + \cos^2 2x dx \\ &= \frac{1}{2}\int 1 + 2\cos 2x + \frac{1}{2}(1+\cos 4x) dx \\ &= \frac{1}{2}\int \frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x dx \\ &= \frac{1}{2}\left(\frac{3}{2}x - \sin 2x - \frac{1}{8}\sin 4x\right) + c\end{aligned}$$

Now let's look at integrals containing $\tan x$ and $\sec x$. Consider the integral

$$\int \tan^m x \sec^n x dx$$

1. If m is odd, rearrange the integral to contain $\tan x \sec x dx$, use the identity $\tan^2 x = \sec^2 x - 1$ so that the integral is now in the form $\int f(\sec x) \cdot \tan x \sec x dx$ and use the substitution $u = \sec x$.
2. If n is even, rearrange the integral to contain $\sec^2 x dx$, use the identity $\sec^2 x = \tan^2 x + 1$ so that the integral is now in the form $\int f(\tan x) \cdot \sec^2 x dx$, and the substitution $u = \tan x$.
3. If m is odd and n is even, either of the above methods will work.
4. If m is even and n is odd, use integration by parts.

Example. *Solve the integral*

$$\int \tan^4 x \sec^6 x dx$$

Solution:
For this example m is even and n is even, so we will follow the steps outlined in the second of the above cases.

$$\begin{aligned}
\int \tan^4 x \sec^6 x dx &= \int \tan^4 x \sec^4 x \cdot \sec^2 x dx \\
&= \int \tan^4 x (\sec^2 x)^2 \cdot \sec^2 x dx \\
&= \int \tan^4 x (\tan^2 x + 1)^2 \cdot \sec^2 x dx \\
&= \int \tan^4 x (\tan^4 x + 2\tan^2 x + 1) \cdot \sec^2 x dx \\
&= \int (\tan^8 x + 2\tan^6 x + \tan^4 x) \cdot \sec^2 x dx
\end{aligned}$$

This question can now be easily solved by using the substitution $u = \tan x$.

Example. *Solve the integral*

$$\int \tan^2 x \sec x dx$$

Solution:
For this problem, m is even and n is odd, so we must use integration by parts.

$$\int \tan^2 x \sec x dx = \int \tan x \cdot \tan x \sec x dx$$

$$\begin{aligned} u &= \tan x & dv &= \tan x \sec x dx \\ \implies du &= \sec^2 x dx & \implies v &= \sec x \end{aligned}$$

Letting

$$I = \int \tan^2 x \sec x dx,$$

$$\begin{aligned}
\implies I &= \tan x \sec x - \int \sec x \cdot \sec^2 x dx \\
\implies I &= \tan x \sec x - \int \sec x \cdot (\tan^2 x + 1) dx \\
\implies I &= \tan x \sec x - \int \sec x \tan^2 x dx + \int \sec x dx \\
\implies I &= \tan x \sec x - I + \int \sec x dx \\
\implies 2I &= \tan x \sec x + \int \sec x dx \\
\implies 2I &= \tan x \sec x + \ln|\sec x + \tan x| + c \\
\implies I &= \tfrac{1}{2} (\tan x \sec x + \ln|\sec x + \tan x|) + c,
\end{aligned}$$

so our integral has been solved:

$$\int \tan^2 x \sec x dx = \tfrac{1}{2} (\tan x \sec x + \ln|\sec x + \tan x|) + c$$

Similar strategies hold for $\cot x$ and $\csc x$ – see if you can figure them out. For integrals of the form

$$\int \cot^m x \csc^n x dx$$

write down the four cases for m and n being even and odd, and figure out general steps to solving integrals like this! Come to the MHC if you have trouble doing this and we can help.

Exercises

1. Evaluate each of the following integrals:

   (a) $\int \sin^5(x)\cos^2(x)dx$

   (b) $\int \sin^4(x)dx$

   (c) $\int \cot^5(x)\csc^3(x)dx$

   (d) $\int \frac{\cos^2(x)}{\sin(x)}dx$

   (e) $\int \frac{\tan^3(x)\sec^3(x)}{\sin^2(x)}dx$

   (f) $\int \sqrt{\tan(x)}\sec^4(x)dx$

   (g) $\int \frac{\tan^3(x)}{\sec^4(x)}dx$