Math Help Centre

# Pigeonhole Principle

1. *(Midterm 2, Summer 2016)* Use the pigeonhole principle to show that if 12 numbers are chosen from $\{1, 2, 3, \cdots, 21\}$, then there are at least two whose sum is 22. Be specific as to what are the "pigeonholes" in your answer.

2. *(Midterm 2, Winter 2017)* How many distinct integers do I need to choose from 2 to 150 (inclusive) to guarantee that I've chosen two numbers $a$ and $b$ such that $gcd(a, b) > 1$. Justify your answer.

3. *(Midterm 2, Summer 2017)* There was a family with one father, one mother, and twelve children,

   (a) Can we guarantee that there are at least two family members born in the same month? Briefly explain your reasoning.

   (b) Can we guarantee there at least one member of the family was born in September? Briefly explain your reasoning.

4. *(Midterm 2, Fall 2017)* How large a group of people is required to ensure that at least six have birthdays in the same month? Be specific as to what are the "pigeons" and "pigeonholes" in your answer.

5. *(Final exam, Summer 2016)* Show that if you choose any $n + 1$ numbers from the set $S = \{1, 2, 3, \ldots, 2n\}$, then one of the numbers divides another.

6. *(Final exam, Summer 2017)* Prove that if 151 integers are selected from $S = \{1, 2, 3, \ldots, 300\}$, then the selection must include two integers $x$ and $y$ where either $x|y$ or $y|x$.
   **HINT:** $\forall n \in \mathbb{Z}^+, \exists m, k \in \mathbb{Z}^+, k$ odd, such that $n = 2^m k$.

7. *(Final exam, Fall 2017)* Prove that in any group of 109 people, we are guaranteed that at least 10 people have birthdays in the same month. Be specific as to what are the "pigeons" and "pigeonholes" in your answer.

# Pigeonhole Principle Workshop Solutions

1. Divide the set $S = \{1, 2, 3, \ldots, 21\}$ into subsets in the following way:

$\{1, 21\}, \{2, 20\}, \{3, 19\}, \ldots, \{10, 12\}, \{11\}$.

The above subsets are the "pigeonholes".

There are 11 pigeonholes, from which 12 numbers must be chosen, so by the P.P., there must be at least one pigeonhole from which two numbers are chosen

In each set, the numbers add up to 22 (except the last set, but it is impossible for two numbers to be taken from there), so when 12 numbers are chosen, there must be at least two adding to 22.

2. Let's start by noticing that all prime numbers between 2 and 150 will be relatively prime, so their gcd is 1.

So how many primes are there between 2 and 150?

Counting, we find that there are 35 prime numbers between 2 and 150, so we could choose 35 numbers whose gcd is 1.

Now, label all primes $p_1, p_2, \ldots, p_{35}$, and consider the sets

$$S_i = \{ p_i n \mid n \in \mathbb{Z}^+, p_i n \le 150 \}$$

Every number between 2 and 150 is in one of the above sets.

These sets are the pigeonholes, and there are 35. If we choose 36 numbers between 2 and 150, then by the P.P., ~~at~~ at least two of the numbers must belong to the same set.

Take two numbers $a, b \in S_i$

$$\gcd(a,b) = p_i > 1,$$

so if 36 numbers are chosen, there must be at least two whose gcd is greater than 1.

3 (a) Yes.

There are 14 family members (pigeons) and 12 months (pigeonholes). By the P.P. at least two family members must be born in the same month

(b) No.

All members could have been born in January, for example

4. For this questions, the pigeonholes are the months, and the pigeons are the people

There are 12 months, so if each month contained a birthday of 5 people, there would be $12 \times 5 = 60$ people.

One more person guarantees at least one month with 6 birthdays, so 61 people are required

5. The key fact for this question is that every positive integer can be written in the form $2^m k$ where $k$ is odd and $m \in \mathbb{Z}$.

[For example,

$$\begin{array}{lll} 1 = 2^0(1) & 5 = 2^0(5) & 9 = 2^0(9) \\ 2 = 2^1(1) & 6 = 2^1(3) & 10 = 2^1(5) \\ 3 = 2^0(3) & 7 = 2^0(7) & 11 = 2^0(11) \\ 4 = 2^2(1) & 8 = 2^3(1) & 12 = 2^2(3) \\ \vdots & \vdots & \vdots \end{array}$$
]

In the set $S = \{1, 2, 3, \ldots, 2n\}$, half of those numbers are odd. (ie., $n$ odd numbers)

The odd numbers are $\{1, 3, 5, \ldots, (2n-1)\}$.

So for all $k \in \{1, 2, 3, \ldots, (2n-1)\}$, define

$$S_k = \{ n \in S \mid \exists m \in \mathbb{Z}^+, n = 2^m \cdot k \}$$

[For example,

$$\begin{aligned} S_{21} &= \{(1)(21), (2)(21), (4)(21), (8)(21)\} \\ &= \{21, 42, 84, 168\} \end{aligned}$$
]

There are $n$ of the sets $S_k$, (these sets are the pigeonholes), so if we choose $n+1$ numbers (pigeons), there must be at least one set from which two numbers are chosen (say, $S_i$).

That means that we have chosen two numbers $a = 2^{m_1} i$ and $b = 2^{m_2} i$.

If $m_1 > m_2$, then $2^{m_2} i \mid 2^{m_1} i$, so $b \mid a$

6. The approach for this question is the same as the last:

There are 150 odd numbers between 1 and 300: $\{1, 3, 5, \ldots, 299\}$.

Define 150 sets $S_k = \{ n \in S \mid \exists m \in \mathbb{Z}^+ ; n = 2^m k \}$

where $k \in \{1, 3, 5, \ldots, 299\}$

If we select 151 numbers (pigeons) from 150 sets $S_k$ (pigeonholes), there must be at least one set, say $S_i$ from which two numbers are taken, so we have chosen $a = 2^{m_1} i$ and $b = 2^{m_2} i$. If $m_1 > m_2$, then $2^{m_2} i \mid 2^{m_1} i$, so $a \mid b$.

7. In a group of 109 people (pigeons), each having a birthday in one of 12 months (pigeonholes), by the P.P. there must be at least one month with at least

$$\left\lfloor \frac{109}{12} \right\rfloor + 1 = 9+1 = 10$$

birthdays.

This is the floor function. You may have seen this done with the ceiling function:

$$\left\lceil \frac{109}{12} \right\rceil = 10.$$