# Path and circuits

The following is a summary of the different types of paths and circuits you need to know for MATH1010.

**Definition.** A **path** in a graph is a succession of adjacent edges, with no repeated edges, that joins two vertices.

**Definition.** A **circuit** is a path which joins a node to itself.

**Definition.** An **Euler path** in a graph without isolated nodes is a path that contains every edge exactly one.

**Definition.** An **Euler circuit** in a graph without isolated nodes is a circuit that contains every edge exactly one.

**Definition.** An **Hamiltonian circuit** in a graph is a circuit that passes through every node exactly once, but never passes through the starting and ending node.

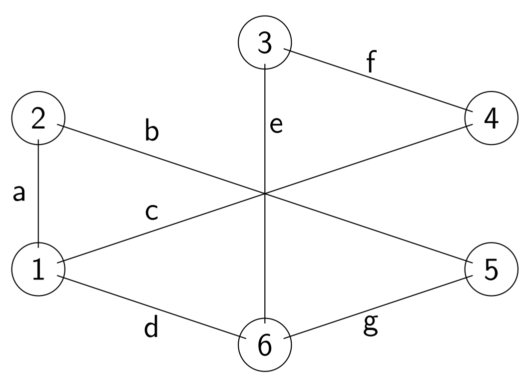
Note that a Hamiltonian circuit need not contain every edge, but it is a type of circuit, so it can not repeat edges.
Euler's first and second theorem are stated here as well for your convenience.

**Theorem** (Euler's First Theorem)**.** *A connected graph has an Euler circuit if and only if the degree of every node is even.*

**Theorem** (Euler's Second Theorem)**.** *A connected graph has an Euler path if and only if the graph has exactly two nodes with odd degree.*

Let's apply these definitions and theorems to the following graphs.

**Example.** Consider the following graph, with nodes $1-6$ and edges $a-g$. Give an example (if it exists) of each of the following.




1. A path
2. A circuit
3. An Euler path
4. An Euler circuit
5. A Hamiltonian circuit

*Solution:*

1. We have many options for paths. For example, here are some paths from node 1 to node 5:
$$a \to b$$
$$d \to g$$
$$c \to f \to e \to g$$
See if you can find all paths from node 6 to node 2.

2. Again, we have a couple of options for circuits. For example, a circuit on node 6:
$$e \to f \to c \to d$$
A circuit on node 2:
$$b \to g \to d \to a$$

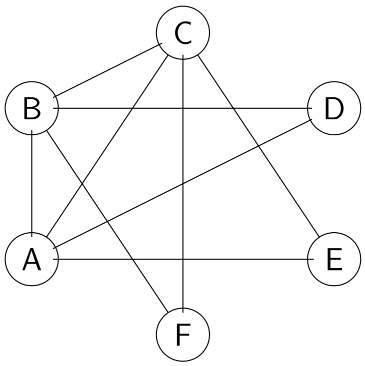
3. First, lets use Euler's second theorem to decide if there is an Euler path. If there is, we will look for one. The degree set of this graph (listing from node 1 to node 6 in ascending order) is:
$$\{3, 2, 2, 2, 2, 3\}$$
According to Euler's second theorem, there is an Euler path if and only if the graph has exactly two nodes with odd degree. This is true for this graph, so we should be able to find one. When searching for an Euler path, you must start on one of the nodes of odd degree and end on the other. Here is an Euler path:
$$d \to e \to f \to c \to a \to b \to g$$

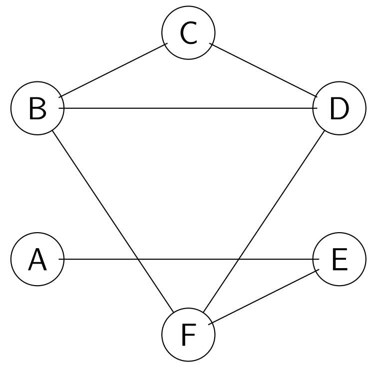
4. Before searching for an Euler circuit, let's use Euler's first theorem to decide if one exists. According to Euler's first theorem, there is an Euler circuit if and only if all nodes have even degree. But from the previous question, we know that we have two nodes of degree 3, so this condition is not satisfied. Hence, no Euler circuit.

5. There are many Hamiltonian circuits here. Starting from node 1, we have
$$a \to b \to g \to e \to f \to c$$
For this Hamiltonian circuit, we did not use edge $d$, but that is totally fine. Remember, a Hamiltonian circuit must use every <u>node</u>, not necessarily every edge.

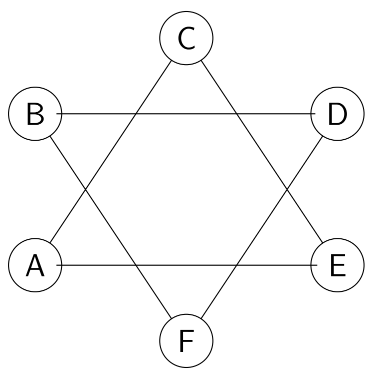
# Exercises

1. For each of the graphs below, give an example of a path, a ciruit, an Euler path, and Euler circuit and a Hamiltonian circuit (if they exist). If they do not exist, explain why.




2. The following graph does not have an Euler circuit. Why not?




*Call in to the Math Help Centre if you are having trouble solving these exercises, or even just to confirm you're doing it right!*