Logic

1. *(Midterm 1, Summer 2016)* Give an implication $p \to q$, write down the statement in symbol form for the:

   (a) Contrapositive.

   (b) Converse.

2. *(Midterm 1, Summer 2016 and Fall 2016)* Negate and simplify the statement $\exists x[p(x) \vee \neg q(x)]$.

3. *(Midterm 1, Summer 2016 and Fall 2016)* Use truth tables to determine whether $\neg p \to \neg(p \wedge q)$ is a tautology. Explain your answer.

4. *(Midterm 1, Fall 2016)* Construct a truth table for the following compound statement:

$$p \to (q \to r)$$

5. *(Midterm 2, Fall 2016)* For any statements p and q, define the connective "Nand" or "Not ...and ..." by

$$p \uparrow q \iff \neg(p \wedge q).$$

   Represent $p \wedge q$ using **only** the Nand connective (that is, your answer should include only the brackets, and the symbols "p", "q" and "$\uparrow$"). Show all your steps. You may use the following without proof:

$$\neg p \iff p \uparrow p \qquad \text{and} \qquad p \to q \iff p \uparrow (q \uparrow p)$$

6. *(Midterm 1, Winter 2017 and Final Exam, Summer 2016)* Negate and simplify the following statement, showing all steps: $\forall x \exists y[xy = xy^2]$.

7. *(Midterm 1, Winter 2017)* For the universe of all integers, let $p(x)$, $q(x)$ and $t(x)$ be the following open statements:

$$p(x) : x > 0, \qquad q(x) : x \text{ is even}, \qquad t(x) : x \text{ is (exactly) divisible by } 5$$

   Write each of the following statements in symbolic form:

   (a) There exists a positive integer that is even.

   (b) No even integer is divisible by 5.

8. *(Midterm 1, Fall 2017)* Use truth tables to prove that the following statement is a tautology:

$$((a \to b) \wedge (b \to a)) \iff (a \leftrightarrow b)$$

9. *(Final Exam, Fall 2016)* Negate the following: $\forall X\, \exists Y\, \exists C\, \forall Z\, p$.

10. *(Final Exam, Fall 2016)* Prove the following argument, stating justification for each step.

$$\begin{array}{ccc} p & \rightarrow & r \\ \neg p & \rightarrow & q \\ q & \rightarrow & s \\ \hline \neg r & \rightarrow & s \end{array}$$

11. *(Final Exam, Fall 2016)* Simplify $(p \vee q) \wedge \neg(\neg p \wedge q)$. Show all your steps.

12. *(Final Exam, Winter 2017)* Let $p(x)$, $q(x)$ denote the following open statements:

$$p(x): \quad x \leq 3 \qquad\qquad q(x): \quad x+1 \text{ is odd}$$

If the universe consists of all integers, circle which of the following are TRUE and cross out the ones that are FALSE:

$$q(1) \qquad\qquad \neg p(3) \qquad\qquad p(7) \vee q(7)$$

$$p(3) \wedge q(4) \qquad\qquad \neg(p(-4) \wedge \neg q(-3))$$

$$\neg p(-4) \wedge \neg q(-3) \qquad\qquad \exists x[p(x) \wedge q(x)] \qquad\qquad \forall x q(x)$$

13. *(Final Exam, Winter 2017)* Construct a truth table for the following compound statement:

$$p \rightarrow (q \rightarrow r)$$

14. For primitive statements p, q, r and s, simplify the compound statement

$$[[[(p \wedge q) \wedge r] \vee [(p \wedge q) \wedge \neg r]] \vee \neg q] \rightarrow s$$

1. (a) Contrapositive : $\neg q \to \neg p$

(b) Converse : $q \to p$.

Note also : Inverse : $\neg p \to \neg q$

2. $\neg \exists x [p(x) \vee \neg q(x)]$

$\Longleftrightarrow \forall x \neg [p(x) \vee \neg q(x)]$

$\Longleftrightarrow \forall x [\neg p(x) \wedge \neg\neg q(x)]$

$\Longleftrightarrow \forall x [\neg p(x) \wedge q(x)]$

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p	q	$\neg p$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \to \neg(p \wedge q)$
0	0	1	0	1	1
0	1	1	0	1	1
1	0	0	0	1	1
1	1	0	1	0	1

It is a tautology because the column for $\neg p \to \neg(p \wedge q)$ contains all 1's.

4.

p	q	r	$q \to r$	$p \to (q \to r)$
0	0	0	1	1
0	0	1	1	1
0	1	0	0	1
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	0	0
1	1	1	1	1

5. $p \wedge q \iff \neg(\neg(p \wedge q))$

$\iff \neg(p \uparrow q)$

$\iff (p \uparrow q) \uparrow (p \uparrow q)$

6. $\neg \forall x \, \exists y \, [xy = xy^2]$

$\iff \exists x \, (\neg \exists y \, [xy = xy^2])$

$\iff \exists x \, \forall y \, \neg [xy = xy^2]$

$\iff \exists x \, \forall y \, [xy \neq xy^2]$

7. (a) $\exists x \left[p(x) \wedge q(x) \right]$

(b) $\forall x \left[\neg \left(q(x) \wedge t(x) \right) \right]$

8 Let $X = \left((a \to b) \wedge (b \to a) \right) \Leftrightarrow (a \leftrightarrow b)$

a	b	$a \to b$	$b \to a$	$(a \to b) \wedge (b \to a)$	$a \leftrightarrow b$	X
0	0	1	1	1	1	1
0	1	1	0	0	0	1
1	0	0	1	0	0	1
1	1	1	1	1	1	1

9. $\neg \forall x \exists y \exists c \forall z \; p$

$\Leftrightarrow \exists x \neg \exists y \exists c \forall z \; p$

$\Leftrightarrow \exists x \forall y \neg \exists c \forall z \; p$

$\Leftrightarrow \exists x \forall y \forall c \neg \forall z \; p$

$\Leftrightarrow \exists x \forall y \forall c \exists z \neg p$

10. $p \to r$ (premise)

$\neg r \to \neg p$ (contrapositive)

$\neg p \to q$ (premise)

$\neg r \to q$ (syllogism)

$q \to s$ (premise)

$\neg r \to s$ (syllogism)

11. $(p \vee q) \wedge \neg(\neg p \wedge q)$

$\Leftrightarrow (p \vee q) \wedge (p \vee \neg q)$

$\Leftrightarrow p \vee (q \wedge \neg q)$

$\Leftrightarrow p \vee F_0$

$\Leftrightarrow p$

12. $p(x): x \leq 3$ $\quad$ $q(x): x+1$ is odd

$\cancel{q(1)}$ $\quad$ $\cancel{\neg p(3)}$ $\quad$ $\cancel{p(7) \vee q(7)}$

$p(3) \wedge q(4)$ $\quad$ $\neg(\cancel{p(-4) \wedge \neg q(-3)})$

$\neg(\cancel{p(-4) \wedge \neg q(-3)})$ $\quad$ $\exists x[p(x) \wedge q(x)]$ $\quad$ $\cancel{\forall x\, q(x)}$

13.

p	q	r	$q \to r$	$p \to (q \to r)$
0	0	0	1	1
0	0	1	1	1
0	1	0	0	1
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	0	0
1	1	1	1	1

10. $\left[\left[\left[(p \wedge q) \wedge r\right] \vee \left[(p \wedge q) \wedge \neg r\right]\right] \vee \neg q\right] \longrightarrow s$

$\Leftrightarrow \left[\left[(p \wedge q) \wedge (r \vee \neg r)\right] \vee \neg q\right] \longrightarrow s$

$\Leftrightarrow \left[\left[(p \wedge q) \wedge T_0\right] \vee \neg q\right] \longrightarrow s$

$\Leftrightarrow \left[(p \wedge q) \vee \neg q\right] \longrightarrow s$

$\Leftrightarrow \left[(p \vee \neg q) \wedge (q \vee \neg q)\right] \longrightarrow s$

$\Leftrightarrow \left[(p \vee \neg q) \wedge T_0\right] \longrightarrow s$

$\Leftrightarrow (p \vee \neg q) \longrightarrow s$