Math Help Centre

Linear Programming

Many real world problems require the optimization of some function subject to a collection of constraints.

Note: Think of "optimizing" as maximizing or minimizing for MATH1010.

For example, company producing furniture wishes to maximize production, but is constrained by the amounts of available raw materials, number of workers, etc. The company wants to find the optimal level of production under these constraints.
When faced with a linear programming problem, the first obstacle you'll face is in setting up the problem – i.e., finding the objective function (a formula for the quantity you wish to maximize) and the constraints (these will be represented by a system of inequalities). Once you have figured these things out, your linear programming problem can be stated mathematically as:

Maximize the function

$$P = Ax + By + C$$

subject to the constraints

$$\begin{gathered} a_1x + b_1y < c_1 \\ a_2x + b_2y < c_2 \\ \vdots \\ a_nx + b_ny < c_n \end{gathered}$$

(The $<$ signs above could be any of the inequality symbols $<, >, \leq, \geq$ depending on the question.) The function $P = Ax + By + C$ is the objective function, and the system of linear inequalities represent the constraints. Once these have been found, the following steps are taken:

1. Draw the feasible set of the constraints.
2. Find the corner points of the feasible set.
3. Test the corner points in the objective function and choose the optimal value.

According to the Fundamental Theorem of Linear Programming (this theorem should be in your notes, but it is stated below for your convenience), the value chosen from the above steps is the optimal solution to the problem.

Theorem. *Fundamental Theorem of Linear Programming:*
If the linear programming problem

$$P = Ax + By + C$$

subject to the constraints

$$\begin{gathered} a_1x + b_1y < c_1 \\ a_2x + b_2y < c_2 \\ \vdots \\ a_nx + b_ny < c_n \end{gathered}$$

has an optimal solution, then it must occur at a corner point of the feasible set. Moreover, if the objective function has the same max or min value at two adjacent corners, then it has the same max or min value at every point on the line segment joining the corners.

Example. *A shipping container can hold two types of crates. Type 1 are small but heavy with volume 2m^3 and weigh 100kg. Type 2 have column 10m^3 and weigh 50kg. The shipping container holds at most 9000kg and 270m^3 of cargo. The company makes a profit of $\$3$ for each type 1 crate and $\$4$ for each type 2 crate shipped.*
How many of each crate maximizes the company's profit, and what is the maximum profit?

Solution: The first thing we need to do is to determine the variables x and y. In questions such as these, there often appear to be many possible variables. Here is a tip if you're struggling to correctly identify the variables in questions like these:

Tip: If you're having trouble identifying the variables, skip ahead to thinking about what the objective function should be.

The thing we wish to maximize is profit. We get \$3 for each type 1 crate and \$4 for each type 2, so our objective function will be

$$P = 3x + 4y$$

where

$$x = \text{Number of crates of type 1}$$
$$y = \text{Number of crates of type 2}$$

Now we need to find the constraints – i.e., the system of inequalities with variables x and y. Very often with linear programming problems, the variables represent physical quantities, so we almost always have the constraints

$$x \geq 0$$
$$y \geq 0$$

We can't ship a negative amount of crates! For the rest of the constraints, a table is helpful:

	Type 1 x	Type 2 y	max/min
Volume	2	10	270
Weight	100	50	9000

As shown above, start with writing the variables x and y along the top, and write the other factors acting as constraints in the left column – in this case, volume and weight of the boxes influence how many can be shipped. The just fill out the information given in the question into the table!

From the table, we can pick out the other constraints:

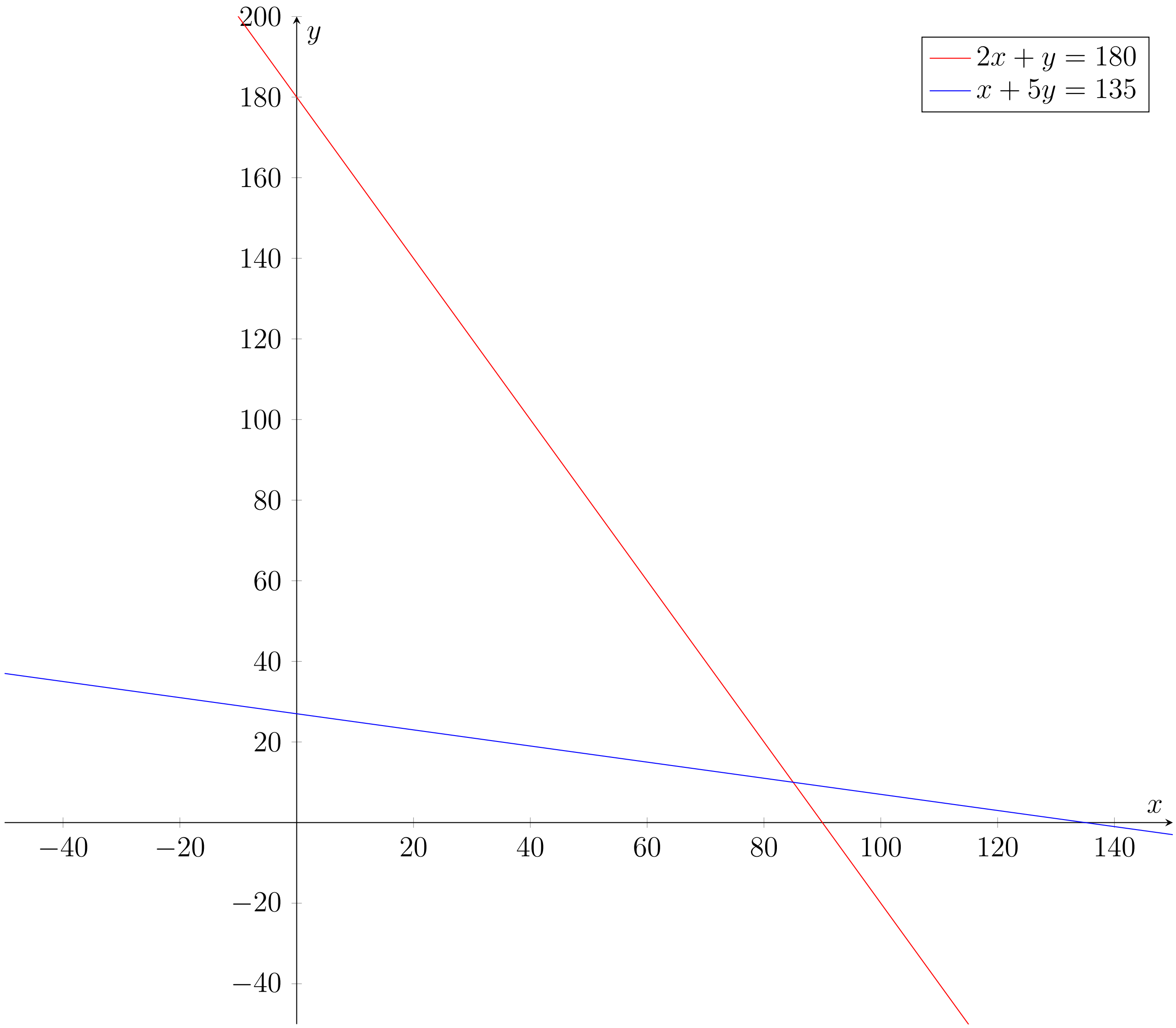
$$2x + 10y \leq 270$$
$$100x + 50y \leq 9000$$

Note that we use $\leq$ for each one because in this particular question, the volume and weight must be **less** than the $270m^3$ and $9000kg$ respectively.

Now all constraints have been found:

$$x + 5y \leq 135$$
$$2x + y \leq 180$$
$$x \geq 0$$
$$y \geq 0$$

The next step in this problem is to draw the feasible set of the system of inequalities. Recall that each inequality is temporarily treated as an equation, and we graph each of the lines on the same xy-axis. Recall, we do this by finding the x and y intercepts of each line – practise this yourself.



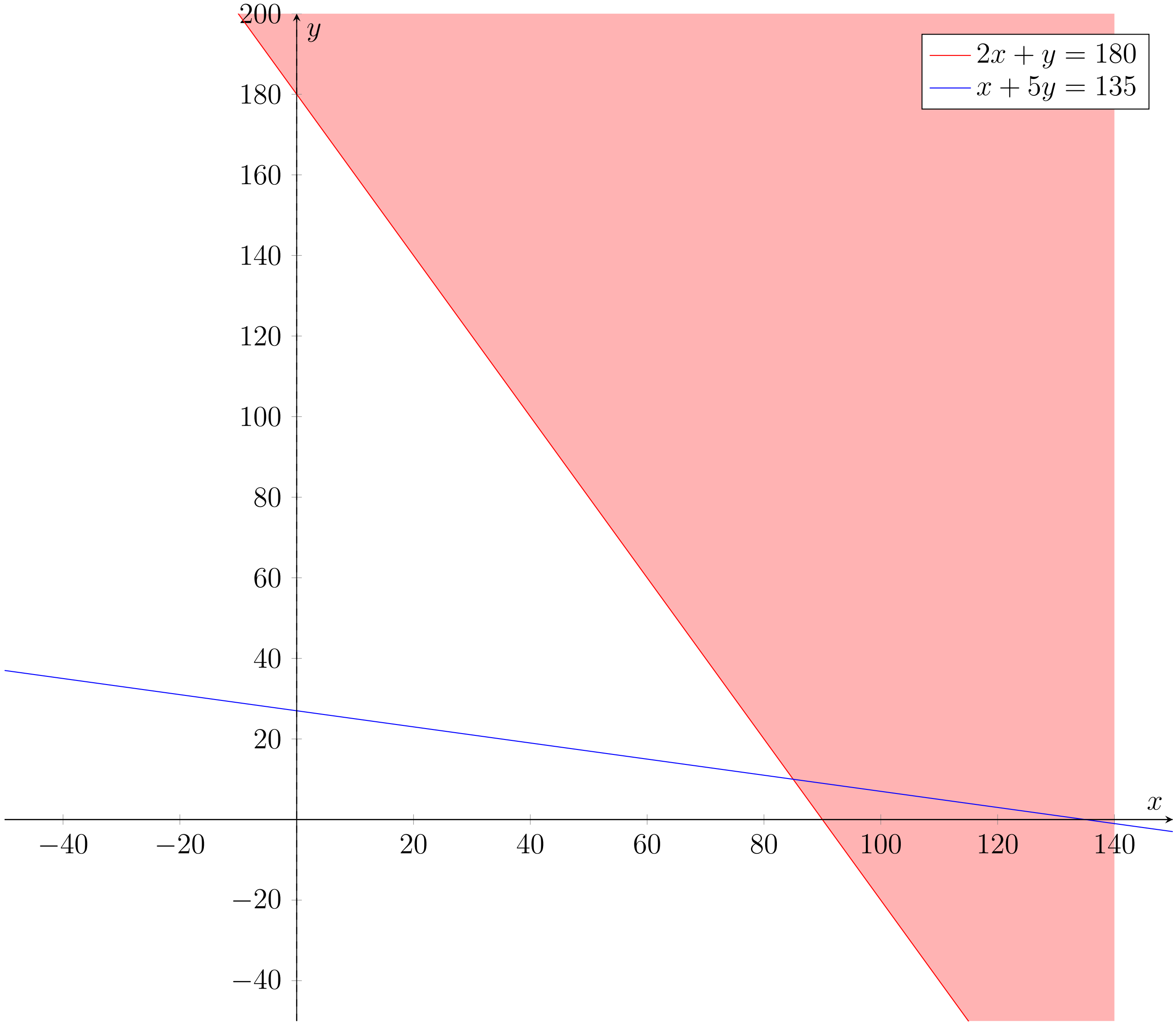
Note that the inequalities $x \geq 0$ and $y \geq 0$ are represented here by the lines $x = 0$, (the y-axis) and $y = 0$, (the x-axis) respectively.

Now that the lines have been drawn in, we need to shade out the appropriate regions in order to leave the feasible set remaining. We do this by testing any point (usually $(0, 0)$ for convenience) in each inequality.

Let's start with the inequality $x + 5y \leq 135$. Testing $(0, 0)$ in this inequality:

$$2(0) + 0 \leq 135 \implies 0 \leq 180,$$

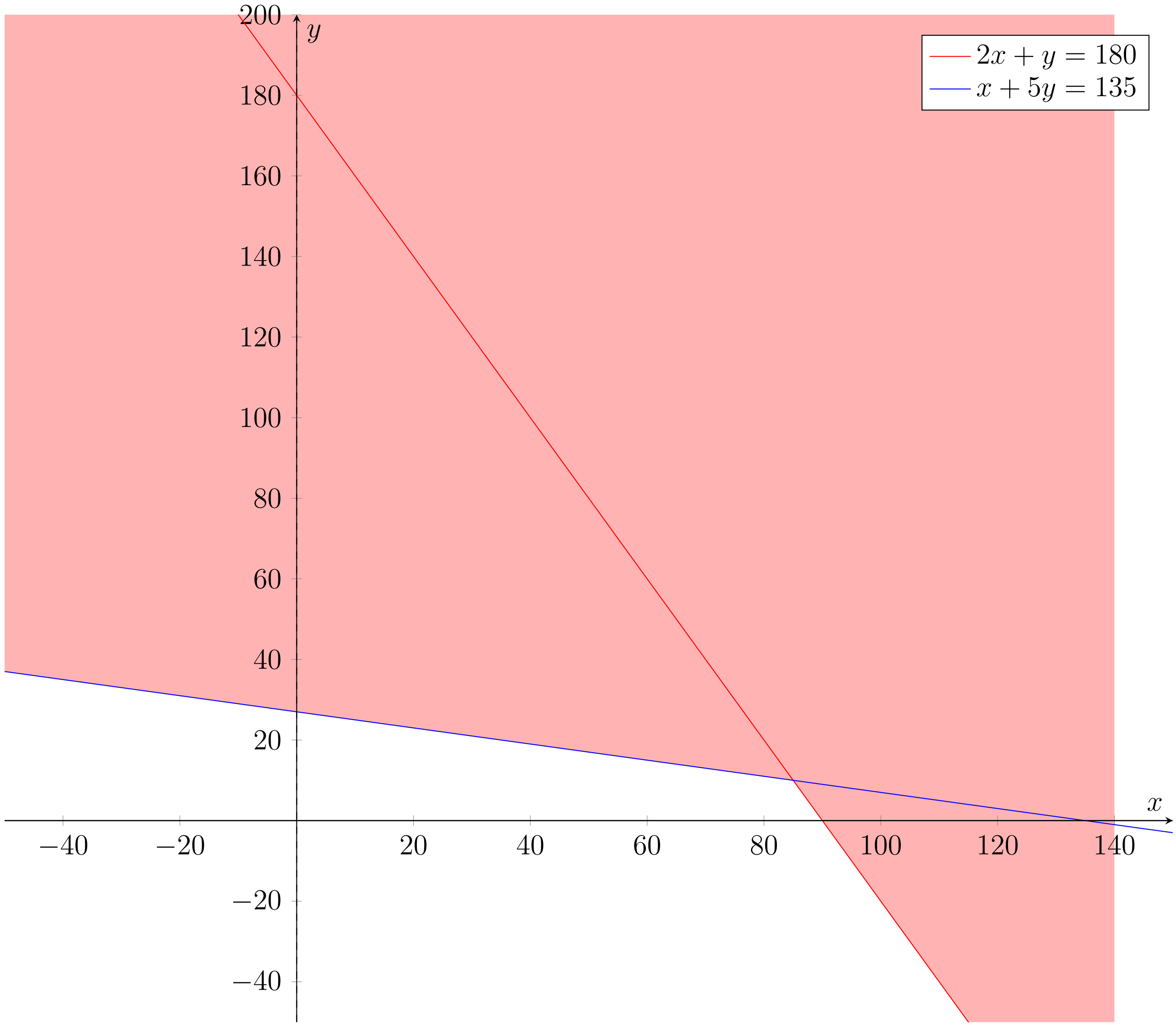
which is true. 0 is less than or equal to 135. Therefore, $(0, 0)$ is in the feasible set of this inequality, so the side of the line $2x + y = 180$ that does NOT contain $(0, 0)$ is shaded OUT.



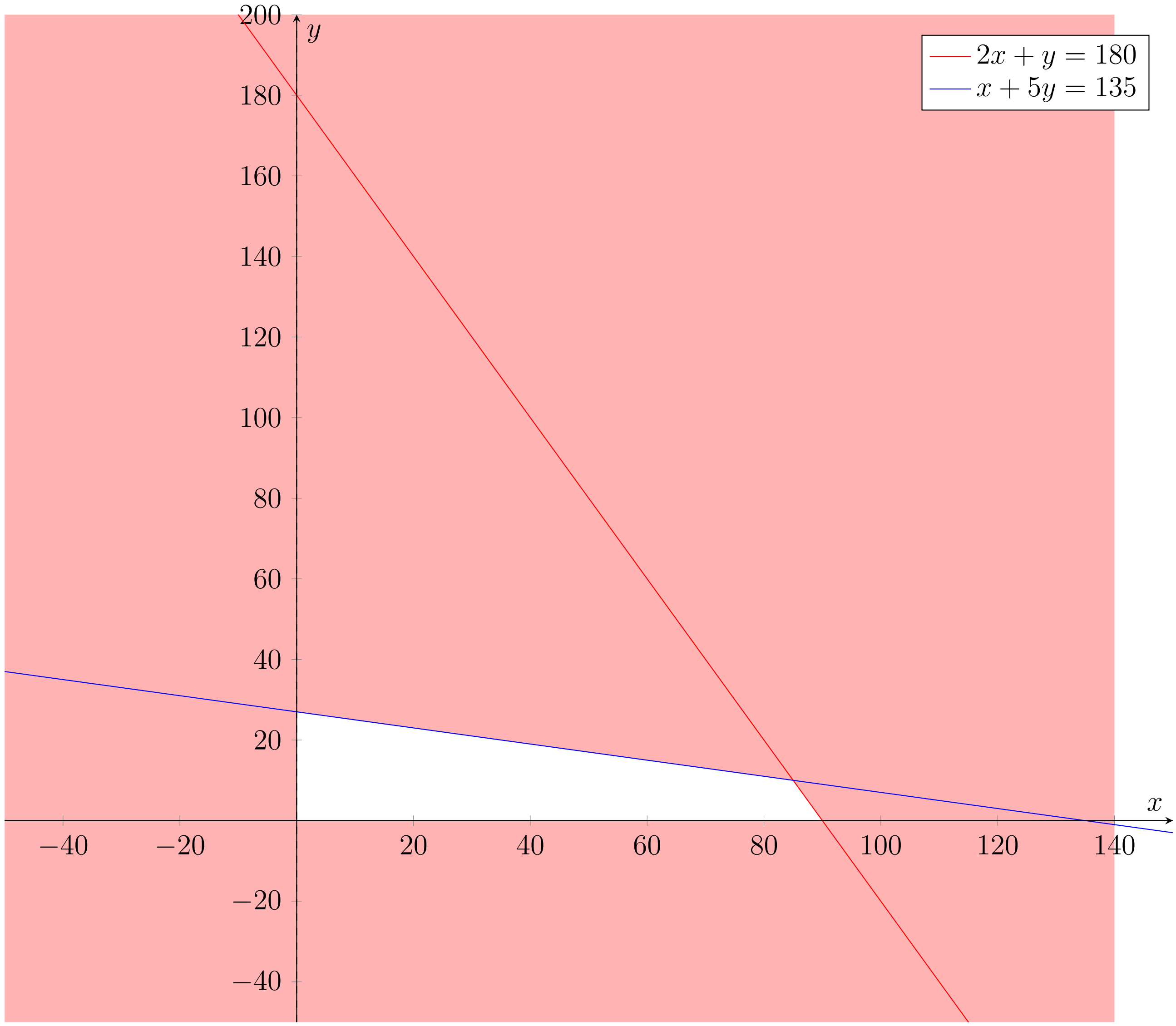
The same approach is taken with the inequality $x + 5y \leq 135$. Testing $(0, 0)$, we find

$$0 + 5(0) \leq 135 \implies 0 \leq 135$$

which is true. Therefore, $(0, 0)$ is in the feasible set of this inequality, so the side of the line $x + 5y = 135$ that does NOT contain $(0, 0)$ is shaded OUT. Our feasible set now becomes



For the final two inequalities $y \geq 0$ and $x \geq 0$, we shade out the regions above the x-axis and to the right of the y-axis respectively, and we are left with the feasible set.



Now that the feasible set has been found, the next step in the process is to find the corner points of the feasible set. Clearly $(0, 0)$ is a corner point of the feasible set. We should also have two other corner points of the feasible set from graphing the lines. Those corner points are $(90, 0)$ and $(0, 27)$.

The final corner point is located at the intersection of the lines $2x + y = 180$ and $x + 5y = 135$. We will find the point using elimination.

$$2x + y = 180 \quad (1)$$

$$x + 5y = 135 \quad (2)$$

Multiplying equation (2) by -2:

$$\begin{array}{rcrcr} 2x & + & y & = & 180 \\ -2x & - & 10y & = & -270 \\ \hline & - & 9y & = & -90, \end{array}$$

so y = 10. Substituting $y = 10$ into (1),

$$\begin{aligned} 2x + y = 180 &\implies 2x + 10 = 180 \\ &\implies 2x = 170 \\ &\implies x = 85, \end{aligned}$$

so the final corner point of the feasible set it $(85, 10)$.

Now that we have all the corner points $(0, 0)$, $(90, 0)$, $(0, 27)$ and $(85, 10)$, all that remains is to test each of them in the objective function $P = 3x + 4y$.

- $(0, 0)$: $P = 3(0) + 4(0) = 0$
- $(90, 0)$: $P = 3(90) + 4(0) = 270$
- $(0, 27)$: $P = 3(0) + 4(27) = 108$
- $(10, 85)$: $P = 3(85) + 4(10) = 295$

Therefore, in order to maximize profit, the company should ship 85 crates of type 1 and 10 crates of type 2, for a maximum profit of $295.