Limits and Continuity

Limits

Calculate the following limits. If the limit does not exist, indicate whether it tends to $+\infty$, $-\infty$, or neither.

1. $\displaystyle\lim_{g\to 4} \frac{2g^2-4}{4+g}$

2. $\displaystyle\lim_{x\to 2} \sqrt{\frac{2x^2+1}{3x-2}}$

3. $\displaystyle\lim_{i\to 1} \frac{i^4-1}{i-1}$

4. $\displaystyle\lim_{x\to 3} \frac{x^3-27}{x-3}$

5. $\displaystyle\lim_{h\to 3} \frac{\frac{1}{h}-\frac{1}{3}}{h-3}$

6. $\displaystyle\lim_{t\to 0} \left(\frac{1}{t}-\frac{1}{t^2+t}\right)$

7. $\displaystyle\lim_{x\to 5^+} \frac{5-x}{|5-x|}$

8. $\displaystyle\lim_{x\to 2^-} \frac{|x-2|}{x^2-4}$

9. $\displaystyle\lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t}$

10. $\displaystyle\lim_{x\to 25} \frac{5-\sqrt{x}}{25x-x^2}$

11. $\displaystyle\lim_{x\to 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}$

12. $\displaystyle\lim_{x\to 0} \frac{\sin(3x)}{x}$

13. $\displaystyle\lim_{x\to 0} \frac{\sin(4x)}{5x}$

14. $\displaystyle\lim_{x\to 0} \frac{\tan(x)}{x}$

15. $\displaystyle\lim_{x\to 0} x^2 \sin(\tfrac{1}{x})$

16. $\displaystyle\lim_{x\to 0} x^3 \cos(\tfrac{1}{x})$

17. $\displaystyle\lim_{x\to 1} (x-1)^2 \sin\left(\frac{1}{x^2-1}\right)$

18. $\displaystyle\lim_{x\to 1^+} \frac{3x}{x-1}$

19. $\displaystyle\lim_{x\to -1^+} \frac{x^2+4x-3}{x+1}$

20. $\displaystyle\lim_{x\to -2^-} \frac{x^3+8}{(x+2)^2}$

21. $\displaystyle\lim_{y\to 1^-} \left(\frac{1}{y-1}-\frac{1}{|y-1|}\right)$

22. $\displaystyle\lim_{x\to \frac{\pi}{2}^+} \tan(x)$

23. $\displaystyle\lim_{x\to -\infty} \frac{4x^2-5x+7}{3x^2-12}$

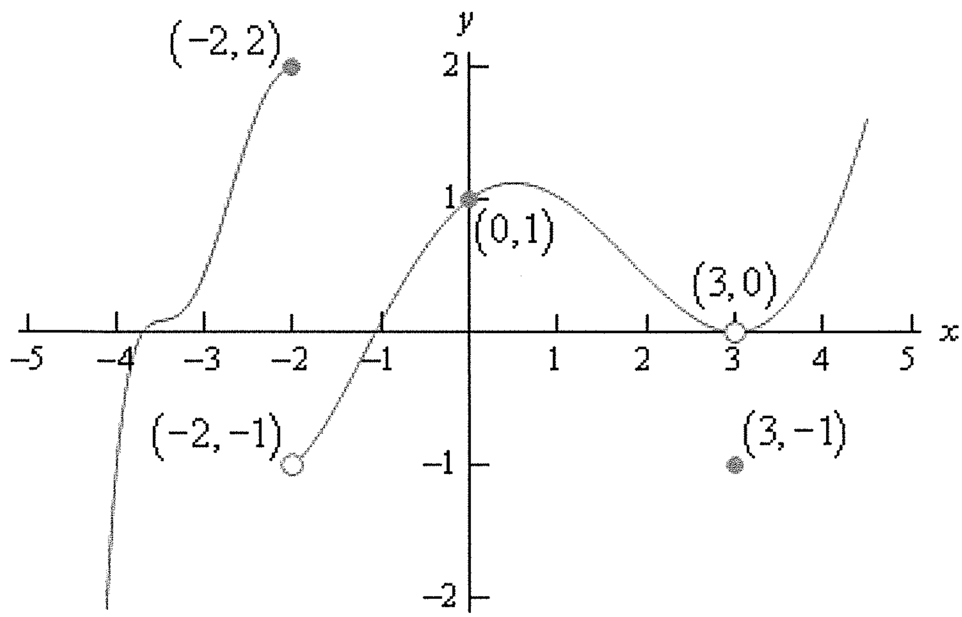
24. $\displaystyle\lim_{x\to \infty} \frac{x^2+2x-4}{3x^3+x^2-5}$

25. $\displaystyle\lim_{x\to -\infty} \frac{\sqrt{2x^2-5x+3}}{x-7}$

26. $\displaystyle\lim_{x\to -\infty} \frac{3x^3+2x+7}{\sqrt{2x^6+4x-3}}$

Continuity

1. Using the graph below, determine if $f(x)$ is continuous at $x = -2, x = 0$ and $x = 3$.



2. Find the values of k for which the function is continuous.

$$f(x) = \begin{cases} 7x - 2, & x \le 1 \\ kx^2, & x > 1 \end{cases}$$

3. Determine the values of a and b that will make the function $f(x)$ continuous at $x = 1$.

$$f(x) = \begin{cases} 2x^2 + 3ax + b, & x < 1 \\ 9, & x = 1 \\ 2bx + a, & x > 1 \end{cases}$$

4. Find the values of x where the function f is discontinuous. Is the function continuous at $x = 0$?

$$f(x) = \begin{cases} x^2, & x < -1 \\ x, & -1 \le x < 1 \\ \frac{1}{x}, & x \ge 1 \end{cases}$$

5. Use the intermediate value theorem to show that the function $f(x) = x^4 - 4x + 2$ has a root between 0 and 1.

6. Use the intermediate value theorem to show that there exists a solution to the equation

$$\cos(\pi x) = \frac{1}{x}.$$

   You do not need to find a solution.

Q1 $\displaystyle\lim_{g\to 4} \frac{2g^2-4}{4+g} = \frac{2(4)^2-4}{4+4} = \frac{32-4}{8} = \frac{28}{8} = \frac{7}{2}.$

Q2 $\displaystyle\lim_{x\to 2} \sqrt{\frac{2x^2+1}{3x-2}} = \sqrt{\frac{2(2)^2+1}{3(2)-2}} = \sqrt{\frac{9}{4}} = \frac{3}{2}.$

Q3 $\displaystyle\lim_{i\to 1} \frac{i^4-1}{i-1} = \lim_{i\to 1} \frac{(i^2)^2-1^2}{i-1}$

$\displaystyle = \lim_{i\to 1} \frac{(i^2-1)(i^2+1)}{i-1}$

$\displaystyle = \lim_{i\to 1} \frac{\cancel{(i-1)}(i+1)(i^2+1)}{\cancel{(i-1)}}$

$\displaystyle = \lim_{i\to 1} (i+1)(i^2+1)$

$= (1+1)(1^2+1)$

$= 4.$

Q4 $\displaystyle\lim_{x\to 3} \frac{x^3-27}{x-3} = \lim_{x\to 3} \frac{\cancel{(x-3)}(x^2+3x+9)}{\cancel{(x-3)}}$

$\displaystyle = \lim_{x\to 3} (x^2+3x+9)$

$= 3^2+3(3)+9 = 27.$

Q5 $\lim_{h\to 3} \frac{\frac{1}{h}-\frac{1}{3}}{h-3} = \lim_{h\to 3} \frac{1}{h-3}\left(\frac{1}{h}-\frac{1}{3}\right)$

$$= \lim_{h\to 3} \frac{1}{h-3}\left(\frac{3-h}{3h}\right)$$

$$= \lim_{h\to 3} \frac{1}{\cancel{(h-3)}} \cdot \frac{-\cancel{(h-3)}}{3h}$$

$$= \lim_{h\to 3} \frac{-1}{3h}$$

$$= \frac{-1}{3(3)} = \frac{-1}{9}.$$

Q6 $\lim_{t\to 0} \left(\frac{1}{t} - \frac{1}{t^2+t}\right) = \lim_{t\to 0}\left(\frac{1}{t} - \frac{1}{t(t+1)}\right)$

$$= \lim_{t\to 0}\left(\frac{t+1-1}{t(t+1)}\right)$$

$$= \lim_{t\to 0} \frac{\cancel{t}}{\cancel{t}(t+1)}$$

$$= \lim_{t\to 0} \frac{1}{t+1}$$

$$= \frac{1}{0+1}$$

$$= 1.$$

Q7 $\displaystyle\lim_{x\to 5^+} \frac{5-x}{|5-x|} = \lim_{x\to 5^+} \frac{5-x}{-(5-x)}$

$\displaystyle = \lim_{x\to 5^+} -1$

$= -1.$

Q8 $\displaystyle\lim_{x\to 2^-} \frac{|x-2|}{x^2-4} = \lim_{x\to 2^-} \frac{-(x-2)}{(x-2)(x+2)}$

$\displaystyle = \lim_{x\to 2^-} \frac{-1}{(x+2)}$

$\displaystyle = \frac{-1}{2+2}$

$\displaystyle = \frac{-1}{4}.$

Q9 $\displaystyle\lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t} \cdot \frac{\sqrt{1+t}+\sqrt{1-t}}{\sqrt{1+t}+\sqrt{1-t}}$

$\displaystyle = \lim_{t\to 0} \frac{(1+t)-(1-t)}{t(\sqrt{1+t}+\sqrt{1-t})} = \lim_{t\to 0} \frac{2\cancel{t}}{\cancel{t}(\sqrt{1+t}+\sqrt{1-t})}$

$\displaystyle = \lim_{t\to 0} \frac{2}{\sqrt{1+t}+\sqrt{1-t}} = \frac{2}{\sqrt{1}+\sqrt{1}} = \frac{2}{2} = 1.$

Q10 $\lim_{x\to 25} \dfrac{5-\sqrt{x}}{25x-x^2} \cdot \dfrac{5+\sqrt{x}}{5+\sqrt{x}}$

$= \lim_{x\to 25} \dfrac{\cancel{(25-x)}}{x\cancel{(25-x)}(5+\sqrt{x})}$

$= \lim_{x\to 25} \dfrac{1}{x(5+\sqrt{x})} = \dfrac{1}{25(5+5)} = \dfrac{1}{250}.$

Q11 $\lim_{x\to 2} \dfrac{\sqrt{6-x}-2}{\sqrt{3-x}-1} \cdot \dfrac{\sqrt{6-x}+2}{\sqrt{6-x}+2}$

$= \lim_{x\to 2} \dfrac{(6-x)-4}{(\sqrt{3-x}-1)(\sqrt{6-x}+2)}$

$= \lim_{x\to 2} \dfrac{(2-x)}{(\sqrt{6-x}+2)(\sqrt{3-x}-1)} \cdot \dfrac{(\sqrt{3-x}+1)}{(\sqrt{3-x}+1)}$

$= \lim_{x\to 2} \dfrac{(2-x)(\sqrt{3-x}+1)}{(\sqrt{6-x}+2)(3-x-1)}$

$= \lim_{x\to 2} \dfrac{\cancel{(2-x)}(\sqrt{3-x}+1)}{(\sqrt{6-x}+2)\cancel{(2-x)}}$

$= \lim_{x\to 2} \dfrac{(\sqrt{3-x}+1)}{(\sqrt{6-x}+2)} = \dfrac{\sqrt{3-2}+1}{\sqrt{6-2}+2} = \dfrac{2}{4} = \dfrac{1}{2}.$

Q12 $$\lim_{x\to 0} \frac{\sin(3x)}{x} \cdot \frac{3}{3} = 3\cdot\lim_{x\to 0} \frac{\sin(3x)}{3x}$$

$$= 3(1)$$

$$= 3.$$

Q13 $$\lim_{x\to 0} \frac{\sin(4x)}{5x} \cdot \frac{\frac{4}{5}}{\frac{4}{5}}$$

$$= \lim_{x\to 0} \frac{4}{5}\frac{\sin(4x)}{4x}$$

$$= \frac{4}{5}\lim_{x\to 0} \frac{\sin(4x)}{4x} = \frac{4}{5}(1) = \frac{4}{5}.$$

Q14 $$\lim_{x\to 0} \frac{\tan(x)}{x} = \lim_{x\to 0} \frac{\sin(x)}{x\cos(x)}$$

$$= \lim_{x\to 0} \frac{1}{\cos(x)} \cdot \frac{\sin(x)}{x}$$

$$= \left(\lim_{x\to 0} \frac{1}{\cos(x)}\right)\left(\lim_{x\to 0} \frac{\sin(x)}{x}\right)$$

$$= \frac{1}{\cos(0)} \cdot (1)$$

$$= 1.$$

Q15 $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$

We know
$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

Multiply through by x^2. Since $x^2 > 0$, this doesn't change the direction of the inequalities:

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

$$\Rightarrow \quad \lim_{x\to 0}(-x^2) \leq \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) \leq \lim_{x\to 0}(x^2)$$

$$\Rightarrow \quad 0 \leq \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) \leq 0$$

$$\Rightarrow \quad \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

Q16 $\lim_{x\to 0} x^3 \cos\left(\frac{1}{x}\right)$.

We know
$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1.$$

If $x > 0$, then $x^3 > 0$, and
$$-x^3 \leq x^3 \cos\left(\frac{1}{x}\right) \leq x^3$$

Q16 (ctd)

So
$$\lim_{x\to 0^+} -x^3 \leq \lim_{x\to 0^+} x^3\cos\left(\frac{1}{x}\right) \leq \lim_{x\to 0^+} x^3$$
$$\underset{0}{\parallel} \qquad\qquad\qquad\qquad \underset{0}{\parallel}$$

$$\Rightarrow \lim_{x\to 0^+} x^3\cos\left(\frac{1}{x}\right) = 0.$$

If $x<0$, then $x^3<0$, so multiplying through by x^3, we have

$$-x^3 \geqslant x^3\cos\left(\frac{1}{x}\right) \geqslant x^3$$

$$\Rightarrow \lim_{x\to 0^-}(-x^3) \geqslant \lim_{x\to 0^-} x^3\cos\left(\frac{1}{x}\right) \geqslant \lim_{x\to 0^-} x^3$$
$$\underset{0}{\parallel} \qquad\qquad\qquad\qquad \underset{0}{\parallel}$$

$$\Rightarrow \lim_{x\to 0^-} x^3\cos\left(\frac{1}{x}\right) = 0$$

Hence $\lim_{x\to 0} x^3\cos\left(\frac{1}{x}\right) = 0.$

Q17 $\lim_{x\to 1} (x-1)^2 \sin\left(\frac{1}{x^2-1}\right)$.

We know

$$-1 \leq \sin\left(\frac{1}{x^2-1}\right) \leq 1$$

Multiply through by $(x-1)^2$, which is positive:

$$-(x-1)^2 \leq (x-1)^2 \sin\left(\frac{1}{x^2-1}\right) \leq (x-1)^2$$

$$\lim_{x\to 1} -(x-1)^2 \leq \lim_{x\to 1} (x-1)^2 \sin\left(\frac{1}{x^2-1}\right) \leq \lim_{x\to 1} (x-1)^2$$

$$0 \leq \lim_{x\to 1} (x-1)^2 \sin\left(\frac{1}{x^2-1}\right) \leq 0$$

Hence $\lim_{x\to 1} (x-1)^2 \sin\left(\frac{1}{x^2-1}\right) = 0.$

Q18 $\lim\limits_{x\to 1^+} \dfrac{3x}{x-1}$ $\begin{matrix}\nearrow 3 \\ \searrow 0^+\end{matrix}$ $= +\infty$

Q19 $\lim\limits_{x\to -1^+} \dfrac{x^2+4x-3}{x+1} \to \dfrac{-6}{0^+} \to -\infty$

Q20 $\lim\limits_{x\to -2^-} \dfrac{x^3+8}{(x+2)^2} = \lim\limits_{x\to -2^-} \dfrac{(x+2)(x^2-4x+4)}{(x+2)^2}$

$= \lim\limits_{x\to -2^-} \dfrac{(x^2-4x+4)}{(x+2)} \to \dfrac{16}{0^-}$

$= -\infty.$

Q21 $\lim\limits_{y\to 1^-} \left(\dfrac{1}{y-1} - \dfrac{1}{|y-1|}\right) = \lim\limits_{y\to 1^-} \left(\dfrac{1}{y-1} + \dfrac{1}{y-1}\right)$

$= \lim\limits_{y\to 1^-} \left(\dfrac{2}{y-1}\right) \to \dfrac{2}{0^-}$

$= -\infty.$

Q22 $\lim\limits_{x\to \frac{\pi}{2}^+} \tan(x) = \lim\limits_{x\to \frac{\pi}{2}^+} \dfrac{\sin(x)}{\cos(x)} \to \dfrac{1}{0^-}$

$= -\infty$ (or look at the graph).

Q23

$$\lim_{x \to -\infty} \frac{4x^2 - 5x + 7}{3x^2 - 12} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$= \lim_{x \to -\infty} \frac{\frac{4x^2}{x^2} - \frac{5x}{x^2} + \frac{7}{x^2}}{\frac{3x^2}{x^2} - \frac{12}{x^2}}$$

$$= \lim_{x \to -\infty} \frac{4 - \underbrace{\frac{5}{x}}_{\to 0} + \underbrace{\frac{7}{x^2}}_{\to 0}}{3 - \underbrace{\frac{12}{x^2}}_{\to 0}}$$

$$= \frac{4}{3}.$$

Q24

$$\lim_{x \to \infty} \frac{x^2 + 2x - 4}{3x^3 + x^2 - 5} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

$$= \lim_{x \to \infty} \frac{\frac{x^2}{x^3} + \frac{2x}{x^3} - \frac{4}{x^3}}{\frac{3x^3}{x^3} + \frac{x^2}{x^3} - \frac{5}{x^3}}$$

$$= \lim_{x \to \infty} \frac{\underbrace{\frac{1}{x}}_{\to 0} + \underbrace{\frac{2}{x^2}}_{\to 0} - \underbrace{\frac{4}{x^3}}_{\to 0}}{3 + \underbrace{\frac{1}{x}}_{\to 0} - \underbrace{\frac{5}{x^3}}_{\to 0}}$$

$$= \frac{0}{3}$$

$$= 0.$$

Q25

$$\lim_{x \to -\infty} \frac{\sqrt{2x^2-5x+3}}{x-7}.$$

$$= \lim_{x \to -\infty} \frac{\frac{1}{x}\sqrt{2x^2-5x+3}}{\frac{1}{x}(x-7)}$$

$$= \lim_{x \to -\infty} \frac{-\sqrt{\frac{1}{x^2}}\sqrt{2x^2-5x+3}}{\left(1-\frac{7}{x}\right)}$$

$$= \lim_{x \to -\infty} \frac{-\sqrt{2-\frac{5}{x}+\frac{3}{x^2}}}{1-\frac{7}{x}}$$

$0 \quad 0$

$\to 0$

$$= \frac{-\sqrt{2}}{1}$$

$$= -\sqrt{2}.$$

Q26

$$\lim_{x \to -\infty} \frac{3x^3+2x+7}{\sqrt{2x^6+4x-3}}$$

$$= \lim_{x \to -\infty} \frac{\frac{1}{x^3}(3x^3+2x+7)}{\frac{1}{x^3}\sqrt{2x^6+4x-3}}$$

$$= \lim_{x \to -\infty} \frac{\left(3+\frac{2}{x^2}+\frac{7}{x^3}\right)}{-\sqrt{\frac{1}{x^6}(2x^6+4x-3)}}$$

$$= \lim_{x \to -\infty} \frac{3 + \frac{2}{x^2} + \frac{7}{x^3}}{-\sqrt{2 + \frac{4}{x^5} - \frac{3}{x^6}}}$$

$$= \frac{3}{-\sqrt{2}}$$

Continuity

Q1

- $f(x)$ is not continuous at $x=-2$

  since $\lim_{x \to -2^-} f(x) = 2$

  and $\lim_{x \to -2^+} f(x) = -1$, and they are not equal.

- $f(x)$ is continuous at $x=0$,

  because $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 1$,

  and $f(0)=1$.

- $f(x)$ is not continuous at $x=3$,

  because $\lim_{x \to 3} f(x) = 0$,

  but $f(3) = -1$, not 0.

Q2 For any value of k, $7x-2$ and kx^2 are continuous on their domains. We only need to ensure continuity at $x=1$.

Need $\underset{x\to 1^-}{\lim} f(x)$ ① $= \underset{x\to 1^+}{\lim} f(x)$ ② $= f(1)$. ③

① $\displaystyle\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (7x-2) = 7(1)-2 = 5.$

② $\displaystyle\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (kx^2) = k(1)^2 = k.$

③ $f(1) = 5.$

$\Rightarrow$ k must equal 5.

Q3 $$f(x) = \begin{cases} 2x^2 + 3ax + b, & x < 1 \\ 9, & x = 1 \\ 2bx + a, & x > 1. \end{cases}$$

Need

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1).$$
$$\quad (1) \qquad\qquad (2) \qquad\qquad (3)$$

(1) $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x^2 + 3ax + b) = 2 + 3a + b.$

(2) $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2bx + a) = 2b + a$

(3) $f(1) = 9.$

So $2 + 3a + b = 9$

and $2b + a = 9. \Rightarrow a = 9 - 2b$

So $2 + 3a + b = 9$ becomes

$$2 + 3(9 - 2b) + b = 9$$
$$27 - 6b + b = 7$$
$$-5b = -20$$
$$\boxed{b = 4}$$

So $\boxed{a = 9 - 2(4) = 1}.$

Q4 $f(x) = \begin{cases} x^2 & x < -1 \\ x & -1 \leq x < 1 \\ \frac{1}{x} & x \geq 1. \end{cases}$

The potential places that f could be discontinuo[us] are at $x = -1$, and $x = 1$. ("break points")

Need to check

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) = f(-1)$$

and

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1).$$

Q5 $f(x) = x^4 - 4x + 2$ is continuous on the interval $[0,1]$, so IVT applies.

* $f(0) = 0^4 - 4(0) + 2 = 2 > 0$
* $f(1) = 1^4 - 4(1) + 2 = -1 < 0$

Hence by IVT, there is some c in $(0,1)$ so that $f(c) = 0$.

i.e. f has a root in $(0,1)$.

Q6 Let $f(x) = \cos(\pi x) - \frac{1}{x}$.

We want to find an interval $[a,b]$ where $f(x)$ is continuous (so it can't contain 0) and then show that $f(a) < 0$ and $f(b) > 0$ or vice versa.

e.g. $[1,2]$

$f(1) = \cos(\pi) - 1 = -1 - 1 = -2 < 0$

$f(2) = \cos(2\pi) - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2} > 0$

So $f(x)$ has a root in $[1,2]$ by IVT.