Limits

Consider the function

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

This function is not defined for $x = 1$, but if we examine the value of f for numbers close to 1, we can observe something interesting:

x	0	0.5	0.9	0.999	1	1.001	1.1	1.5	2
f(x)	1	1.5	1.9	1.999	D.N.E.	2.001	2.1	2.5	3

As x approaches 1 from the left, the $f(x)$ approaches 2. Furthermore, as x approaches 1 from the right, the function also approaches 2. Mathematically, we write this as

$$\lim_{x \to 1^-} \frac{x^2 - 1}{x - 1} = 2 \qquad \text{and} \qquad \lim_{x \to 1^+} \frac{x^2 - 1}{x - 1} = 2$$

respectively.

Since the $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$, we say that the limit of $f(x)$ as x approaches 1 exists, and equals 2. i.e.,

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

Note: The limit only exists provided the left and right limits are equal. For example, suppose we have a function $g(x)$ where

$$\lim_{x \to 1^-} g(x) = 2 \qquad \text{and} \qquad \lim_{x \to 1^+} g(x) = 3.$$

The we would say that the limit of $g(x)$ as x approaches 1 does not exist. i.e.,

$$\lim_{x \to 1} g(x) \text{ D.N.E.}$$

In summary,

$$\boxed{\lim_{x \to c} f(x) = L \iff \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L}$$

To evaluate a limit, we take a different approach to that above – rather than testing the function at values close to the number in question, we use the limit laws (which you should have in your notes). The following are a selection of examples you need to be able to tackle in calculus.

Example. *Evaluate*

$$\lim_{x \to 2} \frac{x}{x - 2} \left(\frac{3}{x + 1} - \frac{1}{x - 1} \right)$$

Solution:

When we try invoking the direct substitution property by substituting 2 into the expression, we get

$$\lim_{x\to 2} \frac{x}{x-2}\left(\frac{3}{x+1} - \frac{1}{x-1}\right) = \frac{2}{2-2}\left(\frac{3}{2+1} - \frac{1}{2-1}\right) = \frac{2}{0}\left(1-1\right)$$

which is not defined, so let's simplify the expression before substituting:

$$\begin{aligned}
\frac{x}{x-2}\left(\frac{3}{x+1} - \frac{1}{x-1}\right) &= \frac{x}{x-2}\left(\frac{3(x-1)}{(x+1)(x-1)} - \frac{1(x+1)}{(x-1)(x+1)}\right) \\
&= \frac{x}{x-2}\left(\frac{3(x-1)-(x+1)}{(x+1)(x-1)}\right) \\
&= \frac{x}{x-2}\left(\frac{3x-3-x-1)}{(x+1)(x-1)}\right) \\
&= \frac{x}{x-2}\left(\frac{2x-4)}{(x+1)(x-1)}\right) \\
&= \frac{x}{x-2}\left(\frac{2(x-2)}{(x+1)(x-1)}\right) \\
&= \frac{2x(x-2)}{(x-2)(x+1)(x-1)} \\
&= \frac{2x\cancel{(x-2)}}{\cancel{(x-2)}(x+1)(x-1)} \\
&= \frac{2x}{(x+1)(x-1)}
\end{aligned}$$

Now let's try the limit again.

$$\begin{aligned}
\lim_{x\to 2} \frac{x}{x-2}\left(\frac{3}{x+1} - \frac{1}{x-1}\right) &= \lim_{x\to 2} \frac{2x}{(x+1)(x-1)} \\
&= \frac{2(2)}{(2+1)(2-1)} \\
&= \frac{4}{(3)(1)} \\
&= \frac{4}{3}
\end{aligned}$$

Example. *Evaluate*

$$\lim_{x\to 1}\frac{x^2-1}{\sqrt{x+1}-\sqrt{x^2+1}}$$

Solution:

Again, the direct substitution property does not immediately work. Substituting 1 into the expression,

$$\lim_{x\to 1}\frac{x^2-1}{\sqrt{x+1}-\sqrt{x^2+1}}=\frac{1-1}{\sqrt{2}-\sqrt{2}}$$

$$=\frac{0}{0}$$

For this question, in order to simplify the expression, it is necessary to multiply above and below by the conjugate of $\sqrt{x+1}-\sqrt{x^2+1}$:

$$\frac{x^2-1}{\sqrt{x+1}-\sqrt{x^2+1}}=\frac{x^2-1}{\sqrt{x+1}-\sqrt{x^2+1}}\cdot\frac{\sqrt{x+1}+\sqrt{x^2+1}}{\sqrt{x+1}+\sqrt{x^2+1}}$$

$$=\frac{(x^2-1)(\sqrt{x+1}+\sqrt{x^2+1})}{(\sqrt{x+1}-\sqrt{x^2+1})(\sqrt{x+1}+\sqrt{x^2+1})}$$

$$=\frac{(x^2-1)(\sqrt{x+1}+\sqrt{x^2+1})}{x+1-(x^2+1)}$$

$$=\frac{(x^2-1)(\sqrt{x+1}+\sqrt{x^2+1})}{x-x^2}$$

$$=\frac{(x^2-1)(\sqrt{x+1}+\sqrt{x^2+1})}{x(1-x)}$$

$$=\frac{(x-1)(x+1)(\sqrt{x+1}+\sqrt{x^2+1})}{x(1-x)}$$

$$=\frac{(x-1)(x+1)(\sqrt{x+1}+\sqrt{x^2+1})}{-x(x-1)}$$

$$=\frac{\cancel{(x-1)}(x+1)(\sqrt{x+1}+\sqrt{x^2+1})}{-x\cancel{(x-1)}}$$

$$=\frac{(x+1)(\sqrt{x+1}+\sqrt{x^2+1})}{-x}$$

Now that a cancellation has been made, let's try the limit again!

$$\lim_{x\to 1}\frac{x^2-1}{\sqrt{x+1}-\sqrt{x^2+1}} = \lim_{x\to 1}\frac{(x+1)(\sqrt{x+1}+\sqrt{x^2+1})}{-x}$$

$$= \frac{(1+1)(\sqrt{1+1}+\sqrt{1^2+1})}{-1}$$

$$= \frac{(2)(\sqrt{2}+\sqrt{2})}{-1}$$

$$= -4\sqrt{2}$$

Example. *Evaluate*

$$\lim_{x\to 4}\frac{x-1}{x^2-5x+4}$$

Solution:

$$\lim_{x\to 4}\frac{x-1}{x^2-5x+4} = \lim_{x\to 4}\frac{x-1}{(x-1)(x-4)}$$

$$= \lim_{x\to 4}\frac{1}{x-4}$$

which is not defined. When no further simplification is possible and a limit is still undefined, we have to examine the left and right limits. Let's start with the left limit.
Try to imagine numbers getting closer and closer to 4 from the left (i.e., 3.9, 3.9999 etc.) being substituted into the expression. The denominator is getting smaller and smaller while the numerator is remaining constant, so the size of the expression is getting extremely large. But since 4 is being approached from the left, the denominator is negative, while the numerator remains positive. Thus, the expression is negative, so

$$\lim_{x\to 4-}\frac{1}{x-4} = -\infty$$

By a similar analysis, we find that

$$\lim_{x\to 4+}\frac{1}{x-4} = \infty$$

Since the limits are not equal,

$$\lim_{x\to 4}\frac{1}{x-4} \text{ D.N.E.}$$

Note: If both limits turned out to be ∞, this does not mean the limit exists. It is still a non-existent limit, except we write

$$\lim_{x\to 4}\frac{1}{x-4} = \infty$$

as it is better describes the nature of the function around $x = 4$ than just "D.N.E."

Example. *Evaluate*

$$\lim_{x\to\infty} \frac{7x^3 - 4x}{3x^3 + x^2 - 2}$$

Solution:

Remember, when taking the limit of a rational expression to infinity, your approach should be to divide the numerate and the denominator by x^a, where a is the highest power of x **in the denominator**.

$$\lim_{x\to\infty} \frac{7x^3 - 4x}{3x^3 + x^2 - 2} = \lim_{x\to\infty} \frac{\frac{7x^3}{x^3} - \frac{4x}{x^3}}{\frac{3x^3}{x^3} + \frac{x^2}{x^3} - \frac{2}{x^3}}$$

$$= \lim_{x\to\infty} \frac{7 - \frac{4}{x^2}}{3 + \frac{1}{x} - \frac{2}{x^3}}$$

As $x \to \infty$, $\dfrac{4}{x^2}$, $\dfrac{1}{x}$ and $\dfrac{2}{x^3}$ go to zero, so

$$\lim_{x\to\infty} \frac{7x^3 - 4x}{3x^3 + x^2 - 2} = \frac{7}{3}$$

Example. *Evaluate*

$$\lim_{x\to-\infty} \frac{4x + 7}{\sqrt{4x^2 + 3}}$$

Solution:

Again, we want to divide the numerator and denominator by x^a where a is the highest power of x in the denominator. Note that x^2 is not the highest power of x in the denominator. We proceed as follows.

$$\lim_{x\to-\infty} \frac{4x + 7}{\sqrt{4x^2 + 3}} = \lim_{x\to-\infty} \frac{4x + 7}{\sqrt{x^2(4 + \frac{3}{x^2})}}$$

$$= \lim_{x\to-\infty} \frac{4x + 7}{\sqrt{x^2}\sqrt{4 + \frac{3}{x^2}}}$$

Now recall the definition of $\sqrt{x^2}$:

$$\sqrt{x^2} = |x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

Since we are taking the limit as $x \to -\infty$, $\sqrt{x^2} = -x$.

$$\lim_{x\to-\infty}\frac{4x+7}{\sqrt{x^2}\sqrt{4+\frac{3}{x^2}}}=\lim_{x\to-\infty}\frac{4x+7}{-x\sqrt{4+\frac{3}{x^2}}}$$

Now we can see that the highest power of x in the denominator is x, so we divide the numerator and denominator by x!

$$\lim_{x\to-\infty}\frac{4x+7}{-x\sqrt{(4+\frac{3}{x^2})}}=\lim_{x\to-\infty}\frac{\frac{4x}{x}+\frac{7}{x}}{\frac{-x\sqrt{4+\frac{3}{x^2}}}{x}}$$

$$=\lim_{x\to-\infty}\frac{\frac{4\cancel{x}}{\cancel{x}}+\frac{7}{x}}{\frac{-\cancel{x}\sqrt{4+\frac{3}{x^2}}}{\cancel{x}}}$$

$$=\lim_{x\to-\infty}\frac{4+\frac{7}{x}}{-\sqrt{4+\frac{3}{x^2}}}$$

$$=\frac{4+0}{-\sqrt{4+0}}$$

$$=-2$$

Exercises

Evaluate the following limits (if they exist):

1. $\lim_{x \to 2} (10x^3 + 4x - 5)$
2. $\lim_{u \to 1} (u^4 - 3u)(u^2 + 5u + 3)$
3. $\lim_{g \to 4} \frac{2g^2 - 4}{4 + g}$
4. $\lim_{x \to 2} \sqrt{\frac{2x^2 + 1}{3x - 2}}$
5. $\lim_{i \to 1} \frac{i^4 - 1}{i - 1}$
6. $\lim_{x \to 1^-} \sqrt{x - 1}$
7. $\lim_{h \to 3} \frac{\frac{1}{h} - \frac{1}{3}}{h - 3}$
8. $\lim_{x \to 2} \frac{|x - 2|}{x^2 - 4}$
9. $\lim_{t \to 0} \frac{\sqrt{1 + t} - \sqrt{1 - t}}{t}$

The following four statements below are false. Provide a counterexample (it can be an equation or a graph) for each.

- A function never reaches a limit value (i.e. if the limit as x approaches a is equal to L, then the function value at a can not equal L.
- To evaluate the limit for any type of function, you should try to plug in the x-value. If you can plug it in without getting an undefined output, then this is the answer.
- A piecewise function that is made up of two pieces, one on $x < 1$ and the other on $x > 1$ will have an undefined limit as x approaches 1.
- If, for a function $f(x)$, $f(2)$ is undefined, then that means that the limit of $f(x)$ as x approaches 2 is also undefined.