Induction

Mathematical Induction is a technique used to prove certain types of mathematical statements. There are two types of mathematical induction; standard induction and strong induction. Note that MATH1210 students only use standard induction, while MATH1240 students use both.

1 Standard Induction

Example. *Prove that the sum of the first n positive integers is given by the formula*

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Solution:

Let P_n be the statement that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

It is helpful to name your statement P_n when using mathematical induction.

Step 1 Base case: Check P_1 holds.
LHS: Sum of first 1 positive integers is 1.
RHS: $\frac{1(1+1)}{2} = 1$.
LHS = RHS, so base case holds.

Here, the base first case of the formula (the first positive integer is 1) is shown to be true.

Step 2 Induction step:

(i) Assume that for some fixed $k \geq 1$, P_k holds. i.e.,

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}.$$

(ii) It remains to show that P_{k+1} is true. i.e.,

$$1 + 2 + 3 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}.$$

$$\begin{aligned} LHS &= 1 + 2 + 3 + \cdots + k + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \quad \text{(by Step 2(i))} \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} = RHS \end{aligned}$$

Hence, P_{k+1} holds.
Therefore, by the principle of mathematical induction (PMI), the statement is proven true for all $n \geq 1$.

This is the key part of the proof.

- In step 2(i), we assume the statement is true for some k.
- In step 2(ii), we showed that if the statement is true for $n = k$, it is also true for $n = k + 1$.
- In step 1 we verified the statement for the first case: $n = 1$.
  - Since it's true for $n = 1$, $\Longrightarrow$ true for $n = 2$.
  - Now it's true for $n = 2$, $\Longrightarrow$ true for $n = 3$.
  - Now it's true for $n = 3$, $\Longrightarrow$ true for $n = 4$.

  This sequence of implications continues forever, so the statement is true for all n.

In general, your induction proofs should match the following template:

Question: Prove that "____________" holds for all $n \geq c$.

Solution:

Let P_n be the statement that "____________".

Step 1: Base Case ($n = c$):

- Show that the base case holds using mathematics.
- "Hence, P_c holds."

Step 2: Induction Step:

(i) "Assume that for some fixed $k \geq c$, P_k is true. i.e.,"

(Insert assumption statement here.)

(ii) "It remains to show that P_{k+1} is true. i.e.,"

(Insert what you aim to prove here.)"

Proof.

Insert your proof here.

"Hence, P_{k+1} holds." $\square$

"By PMI, P_n is true for all $n \geq c$".

Mathematical induction can be used to prove many different types of problems.

Example. *Prove that* $9n! > 2^{2n}$ *for* $n \geq 5$*.*

Solution:
Let P_n be the statement that $9n! > 2^{2n}$.

Step 1 Base case: Check P_5 holds.
LHS: $9(5!) = 9(5)(4)(3)(2)(1) = 1080$
RHS: $2^{2(5)} = 2^{10} = 1024$
LHS $>$ RHS, so base case holds.

Step 2 Induction step:

(i) Assume that for some fixed $k \geq 5$, P_k holds. i.e.,

$$9k! > 2^{2k}.$$

(ii) It remains to show that P_{k+1} is true. i.e.,

$$9(k+1)! > 2^{2k+2}.$$

$$\begin{aligned} LHS &= 9(k+1)! \\ &= 9(k+1)k! \\ &= (k+1)(9k!) \\ &> (k+1)2^{2k} \\ &> (4)2^{2k} \text{ (since } k \geq 5) \\ &= 2^2 2^{2k} \\ &= 2^{2k+2} = RHS \end{aligned}$$

Hence, P_{k+1} holds.
By PMI, P_n is true for all $n \geq 5$

Example. *Prove that for any positive integer* n*,* $6^n - 1$ *is divisible by* 5*.*

Solution:
Let P_n be the statement that $6^n - 1$ is divisible by 5.

Step 1 Base case: Check P_1 holds.
$6^1 - 1 = 6 - 1 = 5$, which is divisible by 5, so base case holds.

Step 2 Induction step:

(i) Assume that for some fixed $k \geq 1$, P_k holds. i.e.,

$$6^k - 1 \text{ is divisible by } 5.$$

(ii) It remains to show that P_{k+1} is true. i.e.,

$$6^{k+1} - 1 \text{ is divisible by } 5.$$

$$6^{k+1} - 1 = 6^k \cdot 6 - 1$$

In order to use our assumption to in this proof, we need $6^k - 1$ to be present. This can be a little tricky, but here is a method I like: take the 6^k, and add and subtract 1. In doing so, we are not changing the expression, we are just adding zero!

$$\begin{aligned} 6^k \cdot 6 - 1 &= (6^k - 1 + 1) \cdot 6 - 1 \\ &= (6^k - 1) \cdot 6 + (1) \cdot 6 - 1 \\ &= (6^k - 1) \cdot 6 + 6 - 1 \\ &= (6^k - 1) \cdot 6 + 5 \end{aligned}$$

By our assumption, $6^k - 1$ is divisible by 5, so can be written in the form $5m$, $m \in \mathbb{Z}$. Therefore,

$$\begin{aligned} (6^k - 1) \cdot 6 + 5 &= (5m) \cdot 6 + 5 \\ &= 5(6m + 1) \end{aligned}$$

Since $m \in \mathbb{Z}$, $(6m + 1) \in \mathbb{Z}$, and $6^{k+1} - 1$ is divisible by 5.
Hence, P_{k+1} holds.
By PMI, P_n is true for all $n \geq 5$.

Example. *Prove that* $1^2 + 2^2 + 3^2 + \cdots + (2n)^2 = \dfrac{n(2n+1)(4n+1)}{3}$, *for all* $n \geq 1$.

Solution:

Let P_n be the statement that $1^2 + 2^2 + 3^2 + \cdots + (2n)^2 = \dfrac{n(2n+1)(4n+1)}{3}$.

Step 1 Base case: Check P_1 holds.
LHS: $1^1 + 2^2 = 5$
RHS: $\dfrac{1(2(1)+1)(4(1)+1)}{3} = \dfrac{(3)(5)}{3} = 5$
LHS = RHS, so base case holds.

> *Note: The base case for the LHS is found by subbing $n = 1$ into the general term $2n^2$. Subbing in $n = 1$, the LHS for the first case of the formula is the $1^2 + 2^2$. The base case being $n = 1$ does not mean you just take the first term of the sum.*

Step 2 Induction step:

(i) Assume that for some fixed $k \geq 5$, P_k holds. i.e.,

$$1^2 + 2^2 + 3^2 + \cdots + (2k)^2 = \frac{k(2k+1)(4k+1)}{3}.$$

(ii) It remains to show that P_{k+1} is true. i.e.,

$$1^2 + 2^2 + 3^2 + \cdots + (2(k+1))^2 = \frac{(k+1)(2(k+1)+1)(4(k+1)+1)}{3}.$$

$$\begin{aligned}
LHS &= 1^2 + 2^2 + 3^2 + \cdots + (2(k+1))^2 \\
&= 1^2 + 2^2 + 3^2 + \cdots + (2k+2)^2 \\
&= 1^2 + 2^2 + 3^2 + \cdots + (2k)^2 + (2k+1)^2 + (2k+2)^2 \\
&= \frac{k(2k+1)(4k+1)}{3} + (2k+1)^2 + (2k+2)^2 \\
&= \frac{k(2k+1)(4k+1)}{3} + \frac{3(2k+1)^2}{3} + \frac{3(2k+2)^2}{3} \\
&= \frac{k(2k+1)(4k+1)}{3} + \frac{3(4k^2+4k+1)}{3} + \frac{3(4k^2+8k+4)}{3} \\
&= \frac{k(2k+1)(4k+1)}{3} + \frac{12k^2+12k+3}{3} + \frac{12k^2+24k+12}{3} \\
&= \frac{k(2k+1)(4k+1)}{3} + \frac{24k^2+36k+15}{3} \\
&= \frac{(2k^2+k)(4k+1)}{3} + \frac{24k^2+36k+15}{3} \\
&= \frac{8k^3+6k^2+k}{3} + \frac{24k^2+36k+15}{3} \\
&= \frac{8k^3+30k^2+37k+15}{3} \\
RHS &= \frac{(k+1)(2(k+1)+1)(4(k+1)+1)}{3} \\
&= \frac{(k+1)(2k+3)(4k+5)}{3} \\
&= \frac{(2k^2+5k+3)(4k+5)}{3} \\
&= \frac{8k^3+10k^2+20k^2+25k+12k+15}{3} \\
&= \frac{8k^3+30k^2+37k+15}{3}
\end{aligned}$$

$LHS = RHS$, so P_{k+1} holds.
By PMI, P_n is true for all $n \geq 5$.

2 Strong Induction

In standard induction, we show that the base case (n_0) of a statement $S(n)$ holds, and assume the k^{th} case of the statement to be true for some k in order to establish the truth of $S(k+1)$. In strong induction however, it may be necessary to show that the statement holds for more than just base case, and in the induction step, it is necessary to assume the truth of all statements $S(n_0), S(n_0+1), \ldots, S(k-1)$ and $S(k)$ in order to establish the truth of $S(k_1)$.

Example. *Prove that every number greater than or equal to 12 can be written in the form $4a+5b$ where $a, b \in \mathbb{Z}$. (for example, $17 = 3(4)+5$)*

Solution:
Let $P(n)$ be the statement that for all $n \geq 12$, $\exists a, b \in \mathbb{Z}$ such that $n = 4a+5b$.

- Base case:

$$\begin{aligned} 12 &= 3(4) \\ 13 &= 2(4)+5 \\ 14 &= 4+2(5) \\ 15 &= 3(5) \end{aligned}$$

- Induction step:

  (i) Suppose that for some fixed $k \geq 15$ $P(i)$ holds for all $12 \leq i \leq k$.

  (ii) It remains to show that $P(k+1)$ holds.

$$k+1 \geq 16 \implies (k+1)-4 \geq 12$$

From (i), we have that there exists $n, m \in \mathbb{Z}$ such that $(k+1)-4 = 4n+5m$

$$\begin{aligned} &(k+1)-4 = 4n+5m \\ &\implies k+1 = 4n+5m+4 \\ &\implies k+1 = 4(n+1)+5m \end{aligned}$$

Hence, $k+1$ is written in the form $4a+5b$ where $a, b \in \mathbb{Z}$.
By the principle of strong mathematical induction, $P(n)$ is true for all $n \geq 12$.

Example. *A sequence of numbers $a_1, a_2, \ldots,$ is defined by*

$$a_1 = 1, \qquad a_2 = 2, \qquad a_n = a_{n-1} + a_{n-2}, \ n \geq 3$$

Prove that for all $n \geq 1$, $a_n < \left(\frac{7}{4}\right)^n$.

Solution:

Let $P(n)$ be the statement that for all $a_n < \left(\frac{7}{4}\right)^n$.

- Base case:

$$a_1 = 1 < \left(\frac{7}{4}\right)^1$$
$$a_2 = 2 < \left(\frac{7}{4}\right)^2$$

  Both $P(1)$ and $P(2)$ hold, so the case case holds.

- Induction step:

  (i) Suppose that for some fixed $k \geq 2$ $P(k)$ and $P(k-1)$ holds.

  (ii) It remains to show that $P(k+1)$ holds.

$$\begin{aligned}
a_{k+1} &= a_k + a_{k-1} \\
&< \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1} \\
&= \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1} \cdot \frac{\left(\frac{7}{4}\right)}{\left(\frac{7}{4}\right)} \\
&= \left(\frac{7}{4}\right)^k + \frac{\left(\frac{7}{4}\right)^k}{\left(\frac{7}{4}\right)} \\
&= \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^k \cdot \left(\frac{4}{7}\right) \\
&= \left(\frac{7}{4}\right)^k \left(1 + \frac{4}{7}\right) \\
&= \left(\frac{7}{4}\right)^k \left(\frac{11}{7}\right) \\
&< \left(\frac{7}{4}\right)^k \left(\frac{7}{4}\right) \\
&= \left(\frac{7}{4}\right)^{k+1}
\end{aligned}$$

  Hence, $P(k+1)$ holds.
  By the principle of strong mathematical induction, $P(n)$ is true for all $n \geq 1$.