Math Help Centre

# Implicit Differentiation

Implicit differentiation is a technique used when you can not isolate one of the variables in an equation. For example, suppose you need to find $\frac{dy}{dx}$, for the curve described by

$$x^2 + xy + y^2 = 3. \tag{1}$$

Isolating $y$ is impossible here, so this is a case where implicit differentiation is required.

Consider (1), above. It is clear that $y$ depends on $x$ in that if $x$ changes, $y$ must also change for the equation to be satisfied. $y$ depends on $x$, but we are unable to explicitly write down that dependence in the form $y = f(x)$. Nonetheless, when taking the derivative, we can still apply the chain rule – notice the similarity: In each of the examples below, on the left we differentiate an expression where all terms are explicitly written as functions of $x$ on the left, while on the right, the expression is almost identical, except $y$ is representing a function of $x$ implicitly.

- $\frac{d}{dx}(x^2+2)^7 = 7(x^2+2)^6 \cdot \frac{d}{dx}(x^2+2)$
- $\frac{d}{dx}x^2(\sin x) = x^2\frac{d}{dx}\sin x + \sin x\frac{d}{dx}x^2$

- $\frac{d}{dx}(y)^7 = 7(y)^6 \cdot \frac{d}{dx}(y)$
- $\frac{d}{dx}(x^2y) = x^2\frac{d}{dx}(y) + y\frac{d}{dx}x^2$

If you find it difficult dealing differentiating $y$ implicitly, try mentally replacing $y$ with a random function of $x$, and see if you can figure that out. Let's re-visit the above example:

$$x^2 + xy + y^2 = 3$$

If differentiating here is difficult for you, try mentally replacing the $y$'s with some random function of $x$:

$$x^2 + x(\cos x) + (\cos x)^2 = 3$$

In order to differentiate the second of the above terms, we must use the product rule:

$$\frac{d}{dx}x(\cos x) = x\frac{d}{dx}\cos x + \cos x\frac{d}{dx}x$$

Now, simply replace the random function with $y$ and you have your answer:

$$\frac{d}{dx}x(y) = x\frac{d}{dx}y + y\frac{d}{dx}x = x\frac{dy}{dx} + y$$

The same applies to the third term, $(\cos x)^2$. This time the chain rule is applied.

$$\frac{d}{dx}(\cos x)^2 = 2(\cos x)\frac{d}{dx}(\cos x), \quad \text{so} \quad \frac{d}{dx}(y)^2 = 2(y)\frac{d}{dx}(y)$$

**Example 1.** *Find* $\frac{dy}{dx}$ *in each of the following equations.*

1. $\ln(y) - x = y^2$
2. $5xy - 2y^2 = 1$
3. $\frac{y^3}{x-2} - x \cdot 2^y = 6$

*Solutions:*

1.

$$\begin{aligned} \ln(y) - x = y^2 &\implies \frac{d}{dx}\ln(y) - \frac{d}{dx}x = \frac{d}{dx}y^2 \\ &\implies \frac{1}{y}\frac{dy}{dx} - 1 = 2y\frac{dy}{dx} \\ &\implies \frac{1}{y}\frac{dy}{dx} - 2y\frac{dy}{dx} = 1 \\ &\implies \frac{dy}{dx}\left(\frac{1}{y} - 2y\right) = 1 \\ &\implies \frac{dy}{dx} = \frac{1}{\left(\frac{1}{y} - 2y\right)} \end{aligned}$$

2.

$$\begin{aligned} 5xy - 2y^2 = 1 &\implies \frac{d}{dx}5xy - \frac{d}{dx}2y^2 = \frac{d}{dx}1 \\ &\implies 5\left(x\frac{dy}{dx} + y\right) - 2(2y)\frac{dy}{dx} = 0 \\ &\implies 5x\frac{dy}{dx} + 5y - 4y\frac{dy}{dx} = 0 \\ &\implies 5x\frac{dy}{dx} - 4y\frac{dy}{dx} = -5y \\ &\implies \frac{dy}{dx}(5x - 4y) = -5y \\ &\implies \frac{dy}{dx} = \frac{-5y}{(5x - 4y)} \end{aligned}$$

3.

$$\begin{aligned}
\frac{y^3}{x-2} - x\cdot 2^y = 6 &\implies \frac{d}{dx}\left(\frac{y^3}{x-2}\right) - \frac{d}{dx}x\cdot 2^y = \frac{d}{dx}6 \\
&\implies \frac{(x-2)\cdot 3y^2\frac{dy}{dx} - y^3(1)}{(x-2)^2} - \left(x\cdot 2^y \ln 2\frac{dy}{dx} + 2^y\right) = 0 \\
&\implies (x-2)\cdot 3y^2\frac{dy}{dx} - y^3 - (x-2)^2\left(x\cdot 2^y\ln 2\frac{dy}{dx} + 2^y\right) = 0 \\
&\implies (x-2)\cdot 3y^2\frac{dy}{dx} - (x-2)^2\cdot x\cdot 2^y\ln 2\frac{dy}{dx} - (x-2)^2 2^y = y^3 \\
&\implies (x-2)\cdot 3y^2\frac{dy}{dx} - (x-2)^2\cdot x\cdot 2^y\ln 2\frac{dy}{dx} = y^3 + (x-2)^2 2^y \\
&\implies \frac{dy}{dx}\left((x-2)\cdot 3y^2 - (x-2)^2\cdot x\cdot 2^y\ln 2\right) = y^3 + (x-2)^2 2^y \\
&\implies \frac{dy}{dx} = \frac{y^3 + (x-2)^2 2^y}{(x-2)\cdot 3y^2 - (x-2)^2\cdot x\cdot 2^y\ln 2}
\end{aligned}$$

# Exercises

1. Solve for $\frac{dy}{dx}$ for each of the following using implicit differentiation.
   (a) $x^2 + y^3 = 6x$
   (b) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$
   (c) $2xy - y^2 = 3$
   (d) $y^4 + 4y - 3x^3\sin(y) = 2x + 1$
2. For each of the following, find the slope of the curve at the given point.
   (a) $x^2y + y^2x = 2, \qquad (1, -2)$
   (b) $\frac{1}{x^3} + \frac{1}{y^3} = 2, \qquad (1, 1)$
   (c) $12(x^2 + y^2) = 25xy, \qquad (3, 4)$

# Answers

1. (a) $\dfrac{dy}{dx} = \dfrac{6-2x}{3y^2}$

   (b) $\dfrac{dy}{dx} = -\dfrac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}}$

   (c) $\dfrac{dy}{dx} = -\dfrac{y}{x-y}$

   (d) $\dfrac{dy}{dx} = \dfrac{2+9x^2\sin(y)}{4y^3+4-3x^3\cos(y)}$

2. (a) $m = 0$

   (b) $m = -1$

   (c) $m = \frac{4}{3}$