Math Help Centre

Gaussian Elimination

Suppose we are trying to solve the following system of equations

$$\begin{aligned} 2x - 3y + 3z &= 6 \\ x + 2y - z &= 3 \\ x - y + z &= 2 \end{aligned}$$

Some of the things we can do to this system in order to solve it are the following:

- Interchange the order of the equations.
- Multiply any of the equations by a nonzero number.
- Add a multiple of one equation to another.

Instead of dealing with the equations, it is more convenient to use an augmented matrix to solve the system. All the same rules would apply, except the notation and language would change slightly: the system we are solving now looks this:

$$\left[\begin{array}{ccc|c} 2 & -3 & 3 & 6 \\ 1 & 2 & -1 & 3 \\ 1 & -1 & 1 & 2 \end{array}\right],$$

and the operations we can perform are not equation operations any more, but row operations:

Elementary Row Operations:

- Interchange rows.
- Multiply a row by a nonzero constant.
- Add a nonzero multiple of one row to another.

These elementary row operations are the **only** operations we are allowed to do on the augmented matrix in solving the system – for example, we can not add a number across a row because that it something we would never do with an equation.

Let's take a closer look at how these operations parallel each other:

$$\begin{aligned} 2x - 3y + 3z &= 6 \\ x + 2y - z &= 3 \\ x - y + z &= 2 \end{aligned}$$

Interchanging the order of the equations:

$$\begin{aligned} x + 2y - z &= 3 \\ 2x - 3y + 3z &= 6 \\ x - y + z &= 2 \end{aligned}$$

Next, we could add -2 times equation 1 to equation 2:

$$\begin{aligned} x + 2y - z &= 3 \\ -7y + 5z &= 0 \\ x - y + z &= 2 \end{aligned}$$

Likewise, adding -1 times equation 1 to equation 2:

$$\begin{aligned} x + 2y - z &= 3 \\ -7y + 5z &= 0 \\ -3y + 2z &= -1 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 3 & 6 \\ 1 & 2 & -1 & 3 \\ 1 & -1 & 1 & 2 \end{array}\right]$$

Interchange rows:

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & -3 & 3 & 6 \\ 1 & -1 & 1 & 2 \end{array}\right]$$

This is equivalent to adding -2 times row 1 to row 2:

$$\xrightarrow{R_2 \to -2R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & 0 \\ 1 & -1 & 1 & 2 \end{array}\right]$$

This is equivalent to adding -1 times row 1 to row 3:

$$\xrightarrow{R_3 \to -R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & 0 \\ 0 & -3 & 2 & -1 \end{array}\right]$$

Now, we have managed to eliminate x from two of the equations, the problem becomes much easier. On the matrix side, this appears at leaving only one non-zero entry in column 1. From now on, we will leave the system of equations behind, and deal only with the augmented matrix.

The elementary row operations we are choosing to perform on the augmented matrix are chosen with a particular goal in mind: we are trying to get the augmented matrix into **row echelon form (REF)**

Properties of Row Echelon Form (REF):

1. The leftmost nonzero entry of every row is a 1.
2. The leading 1 in each row is to the left of the leading 1 in every row below it.
3. The entries below every leading 1 are all 0.
4. Any row of 0's are below the rows with leading 1's.

Notice that the operations we have chosen to do so far are aiming us towards row echelon form. Let's continue to transform this augments matrix in REF:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & 0 \\ 0 & -3 & 2 & -1 \end{array}\right] \xrightarrow{R_2 \to -2R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -1 & 1 & 2 \\ 0 & -3 & 2 & -1 \end{array}\right]$$

$$\xrightarrow{R_3 \to -3R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & -1 & -7 \end{array}\right]$$

$$\xrightarrow{R_2 \to -R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & -1 & -7 \end{array}\right]$$

$$\xrightarrow{R_3 \to -R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

Notice that the matrix now satisfies all the properties of REF (check each of the properties against to matrix to make sure each one is satisfied).
Now that the matrix is in REF, we convert back to equations to solve the system:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 7 \end{array}\right] \iff \begin{aligned} x + 2y - z &= 3 \\ y - z &= -2 \\ z &= 7 \end{aligned}$$

So we have already solve for z. $z = 7$, so

$$\begin{aligned} &\implies y - z = -2 \\ &\implies y - 7 = -2 \\ &\implies y = 5, \end{aligned}$$

so we have now solved for both y and z. $y = 5, z = 7$, so

$$\begin{aligned} &\implies x + 2y - z = 3 \\ &\implies x + 2(5) - 7 = 3 \\ &\implies x = 0, \end{aligned}$$

so we have solved the system:

$$\boxed{x = 0, \quad y = 5, \quad z = 7}$$

Exercises

1. Using Gaussian elimination, solve the following system of equations:

$$\begin{aligned} 3x - y + 3z &= 0 \\ 4x - 2y + 5z &= -2 \\ 7x - y + 2z &= 3 \end{aligned}$$

2. Solve the following system of equations using Gaussian elimination:

$$\begin{aligned} 2x - y + 4z + w &= -13 \\ x + y - 2z + 3w &= 11 \\ x - 3y + z &= -9 \\ 4y + 2z - w &= 1 \end{aligned}$$