Elementary Matrices

An elementary matrix is a matrix that can be obtained from the identity matrix by one single elementary row operation. Multiplying a matrix A by an elementary matrix E (on the left) causes A to undergo the elementary row operation represented by E.

Example. Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & -1 & 1 \\ 2 & -4 & -1 \end{bmatrix}$. Consider the following elementary row operation on A:

$$\begin{bmatrix} 1 & 0 & -1 \\ 3 & -1 & 1 \\ 2 & -4 & -1 \end{bmatrix} \overset{R_2 \to 5R_1 + R_2}{\longrightarrow} \begin{bmatrix} 1 & 0 & -1 \\ 8 & -1 & -4 \\ 2 & -4 & -1 \end{bmatrix}$$

The elementary matrix corresponding to this elementary row operation is

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice that this matrix is found by performing the same elementary row operation on the identity matrix I. The product EA returns the same result as the elementary row operation on A:

$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 3 & -1 & 1 \\ 2 & -4 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 8 & -1 & -4 \\ 2 & -4 & -1 \end{bmatrix}$$

The same is true for the other elementary row operations.

Every elementary matrix is invertible and the inverse of an elementary matrix is also an elementary matrix. It is found by performing the reverse row operation on the identity matrix. The following are the reverse row operations:

- The reverse of $R_i \to kR_i$ is $R_i \to \frac{1}{k}R_i$.
- The reverse of $R_i \to kR_j + R_i$ is $R_i \to -kR_j + R_i$.
- The reverse of $R_i \leftrightarrow R_j$ is $R_i \leftrightarrow R_j$.

Here are some examples of elementary matrices and their inverses (check each of these by multiplying the matrices. if they are inverses, their production should be I):

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

An important fact about elementary matrices is that if a matrix A is invertible, then it can be written as a product of elementary matrices. This is done by examining the row operations used in finding the inverse of a matrix using the direct method.

Example. Let $A = \begin{bmatrix} 1 & -2 & -7 \\ -2 & 5 & 16 \\ 3 & -6 & -20 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -7 & 1 & 0 & 0 \\ -2 & 5 & 16 & 0 & 1 & 0 \\ 3 & -6 & -20 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to 2R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & -7 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 3 & -6 & -20 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to -3R_1 + R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & -7 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to -2R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & -7 & 1 & 0 & 0 \\ 0 & 1 & 0 & 8 & 1 & -2 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to 7R_3 + R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & -20 & 0 & 7 \\ 0 & 1 & 0 & 8 & 1 & -2 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to 2R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 2 & 3 \\ 0 & 1 & 0 & 8 & 1 & -2 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right],$$

so

$$A^{-1} = \begin{bmatrix} -4 & 2 & 3 \\ 8 & 1 & -2 \\ -3 & 0 & 1 \end{bmatrix}$$

Now that the inverse has been calculated, we examine the row operations used in the calculation. Instead of using row operations to transform A to the identity matrix, we can use the elementary matrices corresponding to each row operation and use matrix multiplication. In total, there were 5 row operations used, so we will have 5 elementary matrices multiplying A in that order:

$$E_5E_4E_3E_2E_1A = I$$

- E_1 is the elementary matrix corresponding to the first elementary row operation.
- E_2 is the elementary matrix corresponding to the second elementary row operation, etc.

$$\begin{aligned} E_5E_4E_3E_2E_1A = I &\implies E_4E_3E_2E_1A = E_5^{-1}I \\ &\implies E_3E_2E_1A = E_4^{-1}E_5^{-1} \\ &\implies E_2E_1A = E_3^{-1}E_4^{-1}E_5^{-1} \\ &\implies E_1A = E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1} \\ &\implies A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}, \end{aligned}$$

All that remains is to write down each of the matrices $E_1^{-1}, \ldots, E_5^{-1}$ explicitly:

- $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \implies E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$
- $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \implies E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}.$
- $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}, \implies E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$
- $E_4 = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \implies E_4^{-1} = \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

- $E_5 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \implies E_5^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

$$\begin{aligned} A &= E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Check this for yourself by multiplying all the matrices out.