Math Help Centre

Eigenvalues and Eigenvectors

An eigenvector $\vec{v}$ of a linear transformation T is a non-zero vector that either does not change direction or reverses direction under the action of the T. i.e.,

$$T(\vec{v}) = \lambda\vec{v}, \qquad \lambda \in \mathbb{R}.$$

Here,

- $\vec{v}$ is the eigenvector.
- λ is the eigenvalue.

Every linear transformation can be represented with a square matrix, and we'll almost always use the matrix associated to the linear transformation when trying to solve for eigenvalues and eigenvectors, so remember

$$\boxed{Av = \lambda v, \qquad v \neq 0}$$

where A is the matrix representing T and v is a column matrix representing $\vec{v}$. When asked to find the eigenvalues and eigenvectors associated with a matrix or transformation, you are being asked to find which λ and v satisfy the above equation.

$$\begin{aligned} Av = \lambda v \quad &\Longleftrightarrow \quad Av = \lambda I v \\ &\Longleftrightarrow \quad Av - \lambda I v = 0 \\ &\Longleftrightarrow \quad (A - \lambda I)v = 0 \end{aligned}$$

Since A is a square matrix, this represents a homogeneous system of equations, which has nontrivial solutions when $|A - \lambda I| = 0$ – and this is how we find the eigenvalues! The eigenvalues of a linear transformation are the solutions to

$$\boxed{|A - \lambda I| = 0}$$

Steps for finding eigenvalues and eigenvectors:

Given a square matrix A,

1. Solve $|A - \lambda I| = 0$, for λ
2. For each eigenvalue found above, solve $(A - \lambda I)v = 0$ for v using elementary row operations on the associated augmented matrix.

Example. *Find the eigenvalues and eigenvectors of the matrix*

$$A = \begin{bmatrix} 5 & -3 & -3 \\ -6 & 8 & 6 \\ 12 & -12 & -10 \end{bmatrix}$$

Solution:
First, we find the eigenvalues:

$$A - \lambda I = \begin{bmatrix} 5 & -3 & -3 \\ -6 & 8 & 6 \\ 12 & -12 & -10 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -3 & -3 \\ -6 & 8 & 6 \\ 12 & -12 & -10 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 5-\lambda & -3 & -3 \\ -6 & 8-\lambda & 6 \\ 12 & -12 & -10-\lambda \end{bmatrix}$$

$$\implies |A - \lambda I| = \begin{vmatrix} 5-\lambda & -3 & -3 \\ -6 & 8-\lambda & 6 \\ 12 & -12 & -10-\lambda \end{vmatrix}$$

$$= (5-\lambda)\left[(8-\lambda)(-10-\lambda)+72\right] - (-3)\left[(-6)(-10-\lambda)-72\right] + (-3)\left[72-(8-\lambda)(12)\right]$$
$$= (5-\lambda)\left[\lambda^2+2\lambda-8\right] - (-3)\left[6\lambda-12\right] + (-3)\left[12\lambda-24\right]$$
$$= (5-\lambda)(\lambda+4)(\lambda-2) + 18(\lambda-2) - 36(\lambda-2)$$
$$= (5-\lambda)(\lambda+4)(\lambda-2) - 18(\lambda-2)$$
$$= (\lambda-2)((5-\lambda)(\lambda+4)-18)$$
$$= (\lambda-2)(-\lambda^2+\lambda+2)$$
$$= -(\lambda-2)(\lambda^2-\lambda-2)$$
$$= -(\lambda-2)(\lambda-2)(\lambda+1)$$
$$= -(\lambda-2)^2(\lambda+1)$$

Setting $|A - \lambda I| = 0$, we find that $\lambda = 2$ is an eigenvalue of multiplicity 2, and $\lambda = -1$ is an eigenvalue of multiplicity 1.

Now that we have found the eigenvalues, we move on to the eigenvectors. Remember, we find the eigenvectors by solving $(A - \lambda I)v = 0$ for v for each of the eigenvalues found above.

For $\lambda = -1$:

$$A - \lambda I = \begin{bmatrix} 6 & -3 & -3 \\ -6 & 9 & 6 \\ 12 & -12 & -9 \end{bmatrix}$$

$$
\left[\begin{array}{ccc|c} 6 & -3 & -3 & 0 \\ -6 & 9 & 6 & 0 \\ 12 & -12 & -9 & 0 \end{array}\right] \xrightarrow{R_2 \to R_1 + R_2} \left[\begin{array}{ccc|c} 6 & -3 & -3 & 0 \\ 0 & 6 & 3 & 0 \\ 12 & -12 & -9 & 0 \end{array}\right]
$$

$$
\xrightarrow{R_3 \to -2R_1 + R_3} \left[\begin{array}{ccc|c} 6 & -3 & -3 & 0 \\ 0 & 6 & 3 & 0 \\ 0 & -6 & -3 & 0 \end{array}\right]
$$

$$
\xrightarrow{R_3 \to R_2 + R_3} \left[\begin{array}{ccc|c} 6 & -3 & -3 & 0 \\ 0 & 6 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

$$
\xrightarrow{R_1 \to \frac{1}{6} R_1} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 6 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

$$
\xrightarrow{R_2 \to \frac{1}{6} R_2} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

$$
\xrightarrow{R_1 \to \frac{1}{2} R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Let $z = t$. Then

$$
\begin{aligned}
y &= -\frac{1}{2} t \\
x &= \frac{1}{4} t,
\end{aligned}
$$

so all eigenvectors corresponding to the eigenvalue $\lambda = -1$ are of the form $t\langle \frac{1}{4}, -\frac{1}{2}, 1\rangle$. We apply the same process to find all eigenvectors associated to the other eigenvalue $\lambda = 2$.

For $\lambda = -1$:

$$
A - \lambda I = \begin{bmatrix} 3 & -3 & -3 \\ -6 & 6 & 6 \\ 12 & -12 & -12 \end{bmatrix},
$$

so we need to solve

$$\left[\begin{array}{ccc|c} 3 & -3 & -3 & 0 \\ -6 & 6 & 6 & 0 \\ 12 & -12 & -12 & 0 \end{array}\right]$$

Applying Gauss-Jordan elimination, we find

$$\left[\begin{array}{ccc|c} 3 & -3 & -3 & 0 \\ -6 & 6 & 6 & 0 \\ 12 & -12 & -12 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

so letting $v_2 = s.$ and $v_3 = t$, for $s, t \in \mathbb{R}$ we get

$$v_1 = s + t.$$

Hence, all eigenvalues associated to the eigenvalue $\lambda = 2$ are of the form $\langle s + t, s, t\rangle$, for any $s, t \in \mathbb{R}$.

Exercises

1. Find all eigenvalues and corresponding eigenvectors for the matrix

$$A = \begin{bmatrix} 7 & 4 & -16 \\ 2 & 5 & -8 \\ 2 & 2 & -5 \end{bmatrix}.$$

2. Show that if λ is an eigenvalue of A with eigenvector $\vec{v}$, then λ^n is an eigenvalue of A^n with eigenvector $\vec{v}$.

   Hint: Use induction.

Answers

1. Eigenvalues: $\lambda = 1, 3$. Note that $\lambda = 3$ has multiplicity 2.
   Eigenvectors: For $\lambda = 1$: $\langle 2t, t, t\rangle$. For $\lambda = 3$: $t\langle -s + 4t, s, t\rangle$.