Determinants

Cofactor expansion is a method used in finding the determinant of a matrix. In order to use cofactor expansion, the concept of minors and cofactors must first be discussed.

Minors:

The minor m_{ij} of the $(i,j)^{th}$ entry of a square matrix $A = (a_{ij})_{n\times n}$ is the determinant of the $(n-1)\times(n-1)$ matrix obtained by deleting the row and column of A containing a_{ij}.

For example, consider the matrix

$$A = \begin{bmatrix} 4 & -2 & 5 \\ -3 & 1 & 2 \\ 1 & -7 & -8 \end{bmatrix}$$

For this matrix, the minor m_{23} is found by deleting the second row and third column of the matrix A and calculating the determinant of the result:

$$m_{23} = \begin{vmatrix} 4 & -2 & 5 \\ -3 & 1 & 2 \\ 1 & -7 & -8 \end{vmatrix} = \begin{vmatrix} 4 & -2 \\ 1 & -7 \end{vmatrix} = (4)(-7) - (-2)(1) = -26$$

Cofactors:

The cofactor c_{ij} of the $(i,j)^{th}$ entry of a square matrix $A = (a_{ij})_{n\times n}$ is

$$c_{ij} = (-1)^{i+j} m_{ij}.$$

For example, using the previous matrix A,

$$c_{23} = (-1)^{2+3} m_{23} = (-1)^5(-26) = 26$$

Cofactor Expansion:

To perform cofactor expansion, pick and row or column of the matrix, multiply each entry of the row/column by it's corresponding cofactor, and take the sum of each of these products.

Example. *Find the determinant of the matrix*

$$A = \begin{bmatrix} 4 & -2 & 5 \\ -3 & 1 & 2 \\ 1 & -7 & -8 \end{bmatrix}$$

using cofactor expansion along

1. *The first row.*
2. *The second column.*

Solution:

1.
$$\begin{vmatrix} 4 & -2 & 5 \\ -3 & 1 & 2 \\ 1 & -7 & -8 \end{vmatrix} = (4)\cdot(-1)^{1+1}\begin{vmatrix} 1 & 2 \\ -7 & -8 \end{vmatrix} + (-2)\cdot(-1)^{1+2}\begin{vmatrix} -3 & 2 \\ 1 & -8 \end{vmatrix} + (5)\cdot(-1)^{1+3}\begin{vmatrix} -3 & 1 \\ 1 & -7 \end{vmatrix}$$
$$= (4)(-1)^2(-8+14) + (-2)(-1)^3(24-2) + (5)(-1)^4(21-1)$$
$$= (4)(6) + (2)(22) + (5)(20) = 168$$

2.
$$\begin{vmatrix} 4 & -2 & 5 \\ -3 & 1 & 2 \\ 1 & -7 & -8 \end{vmatrix} = (-2)\cdot(-1)^{1+2}\begin{vmatrix} -3 & 2 \\ 1 & -8 \end{vmatrix} + (1)\cdot(-1)^{2+2}\begin{vmatrix} 4 & 5 \\ 1 & -8 \end{vmatrix} + (-7)\cdot(-1)^{3+2}\begin{vmatrix} 4 & 5 \\ -3 & 2 \end{vmatrix}$$
$$= (-2)(-1)^3(24-2) + (1)(-1)^4(-32-5) + (-7)(-1)^5(8+15)$$
$$= (2)(22) + (1)(-37) + (7)(23) = 168$$

No matter which row or column you choose to do the expansion, the determinant will be the same.

Here is a list of theorems pertaining to determinants which you may find useful in calculating a determinant.

Theorem. *If a square matrix has a row or column of zeros, then its determinant is zero.*

Example.
$$\begin{vmatrix} 2 & -1 & 0 \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 0$$

Theorem. *The determinant of a lower triangular, upper triangular or diagonal matrix is the product of the diagonal entries.*

Example.
$$\begin{vmatrix} 2 & 0 & 0 \\ 13 & -3 & 0 \\ 9 & 1 & 5 \end{vmatrix} = (2)(-3)(5) = -30$$

Theorem. *If two rows or two columns of a matrix A are interchanged, then the determinant of the new matrix B is $|B| = -|A|$.*

Example.

$$\begin{vmatrix} 1 & -3 & 4 \\ 2 & 9 & -6 \\ -1 & 6 & 12 \end{vmatrix} = - \begin{vmatrix} 2 & 9 & -6 \\ 1 & -3 & 4 \\ -1 & 6 & 12 \end{vmatrix}$$

Theorem. *If all entries in a row or column of a matrix A are multiplied by a number k, then the determinant of the new matrix is $k|A|$.*

This theorem is usually used in factoring a number out of a row or column of a determinant, in order to reduce the size of the numbers inside the determinant and make the calculations easier.

Example.

$$\begin{vmatrix} 1 & -3 & 4 \\ 2 & 9 & -6 \\ -1 & 6 & 12 \end{vmatrix} = (3) \begin{vmatrix} 1 & -1 & 4 \\ 2 & 3 & -6 \\ -1 & 2 & 12 \end{vmatrix} = (3)(2) \begin{vmatrix} 1 & -1 & 2 \\ 2 & 3 & -3 \\ -1 & 2 & 6 \end{vmatrix}$$

Theorem. *If a multiple of one row of a determinant is added to another row, then the determinant of the matrix is unchanged. Likewise, if a multiple of one column of a determinant is added to another column, then the determinant of the matrix is unchanged as well.*

Of all the above theorems, the last one is the most useful in finding a determinant. For example, cofactor expansion on a 5×5 matrix would be very time-consuming, but using the last theorem, we can perform row or column operations on the matrix to create zeros (or even convert the matrix to a diagonal form) which makes the calculation a lot faster.

Example. *Find the determinant of the matrix*

$$A = \begin{bmatrix} 1 & 3 & -2 & 3 & 2 \\ -2 & -4 & 2 & 1 & -1 \\ 5 & 14 & 1 & 2 & -1 \\ 4 & 10 & -2 & -2 & -4 \\ 2 & 4 & 5 & -5 & 1 \end{bmatrix}$$

Solution:

$$\begin{vmatrix} 1 & 3 & -2 & 3 & 2 \\ -2 & -4 & 2 & 1 & -1 \\ 5 & 14 & 1 & 2 & -1 \\ 4 & 10 & -2 & -2 & -4 \\ 2 & 4 & 5 & -5 & 1 \end{vmatrix} \underset{\substack{R_3\to -5R_1+R_3 \\ R_4\to -4R_1+R_4 \\ R_5\to -2R_1+R_5}}{\overset{R_2\to 2R_1+R_2}{=}} \begin{vmatrix} 1 & 3 & -2 & 3 & 2 \\ 0 & 2 & -2 & 7 & 3 \\ 0 & -1 & 11 & -13 & -11 \\ 0 & -2 & 6 & -14 & -12 \\ 0 & -2 & 9 & -11 & -3 \end{vmatrix}$$

$$= (1) \begin{vmatrix} 2 & -2 & 7 & 3 \\ -1 & 11 & -13 & -11 \\ -2 & 6 & -14 & -12 \\ -2 & 9 & -11 & -3 \end{vmatrix}$$

$$\underset{\substack{R_3\to -2R_2+R_3 \\ R_4\to -2R_2+R_4}}{\overset{R_1\to 2R_2+R_1}{=}} \begin{vmatrix} 0 & 20 & -19 & -19 \\ -1 & 11 & -13 & -11 \\ 0 & -16 & 12 & 10 \\ 0 & -13 & 15 & 19 \end{vmatrix}$$

$$= -(-1) \begin{vmatrix} 20 & -19 & -19 \\ -16 & 12 & 10 \\ -13 & 15 & 19 \end{vmatrix}$$

which is more easily calculated. The answer could now computed by using cofactor expansion across and of the rows or columns or we could persist further with the row operations to reduce the matrix further. Using more row operations, we could proceed as follows:

$$\begin{vmatrix} 20 & -19 & -19 \\ -16 & 12 & 10 \\ -13 & 15 & 19 \end{vmatrix} \underset{R_2\to -R_3+R_2}{\overset{R_1\to R_3+R_1}{=}} \begin{vmatrix} 7 & -4 & 0 \\ -3 & -3 & -9 \\ -13 & 15 & 19 \end{vmatrix}$$

$$\overset{R_3\to 2R_1+R_3}{=} \begin{vmatrix} 7 & -4 & 0 \\ -3 & -3 & -9 \\ 1 & 7 & 19 \end{vmatrix}$$

The above steps were taken to reduce the size of the numbers in the matrix, as they were a little too large to mentally calculate the determinant. Finally, using cofactor expansion along row one,

$$\begin{vmatrix} 7 & -4 & 0 \\ -3 & -3 & -9 \\ 1 & 7 & 19 \end{vmatrix} = 7(-57+63) + 4(-57+9) + 0(-21+3)$$
$$= 7(6) + 4(-48)$$
$$= 42 - 192$$
$$= -150$$

Exercises

1. Using the theorems regarding determinants, calculate

$$\begin{vmatrix} 3 & 4 & 3 & 1 & -1 \\ 0 & 1 & 2 & 4 & 2 \\ -3 & -4 & -3 & -1 & 5 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & -2 & 1 & 1 \end{vmatrix}$$

2. Find the determinant of the matrix

$$A = \begin{bmatrix} 4 & 6 & 2 & 2 \\ 8 & 16 & 24 & 32 \\ -3 & -2 & -1 & 1 \\ 2 & 3 & 1 & 2 \end{bmatrix}$$

3. Given that

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = c,$$

what is

$$\begin{vmatrix} a_{21} & a_{22} & a_{23}+a_{21} \\ a_{11} & a_{12} & a_{13}+a_{11} \\ \frac{1}{2}a_{31} & \frac{1}{2}a_{32} & \frac{1}{2}(a_{33}+a_{31}) \end{vmatrix}?$$

Answers:

1. -24
2. -192
3. $-\frac{1}{2}c$