# Derivatives

## Definition of the derivative

For each of the following, calculate $f'(x)$ using the definition of the derivative.

1. $f(x) = x^2 - 3x + 4$
2. $f(x) = \dfrac{3}{x^2}$
3. $f(x) = \sqrt{x+5}$
4. $f(x) = \dfrac{2}{\sqrt{x-2}}$

## Differentiation rules

For each of the following, calculate the derivative. Do not simplify your answer.

1. $f(x) = x^7 + 3x^4 + 5e^x - \frac{2}{x} + \sqrt{x}$
2. $g(t) = \sqrt{t^5} + e^{\pi} + \frac{9}{t^3} + \pi t^4$
3. $y = x^3 e^x$
4. $f(x) = (x^2 + 4x + 3)(5x^5 + x^{\frac{1}{2}})$
5. $f(x) = \dfrac{3x^4 + 2}{x^2 - 2x + 3}$
6. $y = \dfrac{\sqrt[3]{x^2}}{\cos(x) + x^{\frac{5}{7}}}$
7. $h(t) = (t^3 + 3t^2)^5$
8. $g(t) = e^{t^3 - 4t^2 + 2t - 5}$
9. $y = (xe^x + 5x^{10})(e^{3x})$
10. $h(x) = x^4 \sin(\cos(x))$
11. $f(x) = \sin^5(4x^2 + 3x) \cdot \cos(3x + 2)$
12. $f(x) = \sqrt{e^{7x} + 5x^9}$
13. $f(x) = \dfrac{3}{(x^3 + 2x + 9)^{\frac{5}{3}}}$
14. $f(x) = \tan\left(\dfrac{3x^2 - 5}{e^{-x}}\right)$
15. $f(x) = \left(\dfrac{3x^2 + 9x}{x^{3/2} - 7x^5 + e^{2x}}\right)^{12}$
16. $f(x) = \sin(\cos(\tan(x)))$
17. $f(x) = e^{\sin(3x)}$
18. $f(x) = \dfrac{x^3\sqrt{4x^2 - 3}}{x^{\frac{2}{9}} + 4x^{12}}$

# Definition of the Derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Q1 $f(x) = x^2 - 3x + 4$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - 3(x+h) + 4 - (x^2 - 3x + 4)}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 3x - 3h + 4 - x^2 + 3x - 4}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 - 3h}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x + h - 3)}{\cancel{h}}$$

$$= \lim_{h \to 0} 2x + h - 3$$

$$= 2x - 3.$$

Q2 $f(x) = \frac{3}{x^2}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{3}{(x+h)^2} - \frac{3}{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{3}{(x+h)^2} - \frac{3}{x^2}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{3x^2 - 3(x+h)^2}{x^2(x+h)^2}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{3x^2 - 3(x^2+2xh+h^2)}{x^2(x+h)^2}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{3x^2 - 3x^2 - 6xh + 3h^2}{x^2(x+h)^2}\right)$$

$$= \lim_{h \to 0} \frac{1}{\cancel{h}}\left(\frac{\cancel{h}(-6x+3h)}{x^2(x+h)^2}\right)$$

$$= \lim_{h \to 0} \frac{-6x+3h}{x^2(x+h)^2}$$

$$= \frac{-6x+3(0)}{x^2(x+0)^2}$$

$$= \frac{-6x}{x^4} = \frac{-6}{x^3}.$$

<u>Q3</u> $f(x) = \sqrt{x+5}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h+5} - \sqrt{x+5}}{h} \cdot \frac{\sqrt{x+h+5} + \sqrt{x+5}}{\sqrt{x+h+5} + \sqrt{x+5}}$$

$$= \lim_{h\to 0} \frac{(x+h+5) - (x+5)}{h\left(\sqrt{x+h+5} + \sqrt{x+5}\right)}$$

$$= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h+5} + \sqrt{x+5}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h+5} + \sqrt{x+5}}$$

$$= \frac{1}{\sqrt{x+5} + \sqrt{x+5}}$$

$$= \frac{1}{2\sqrt{x+5}}.$$

Q4 $\quad f(x) = \dfrac{2}{\sqrt{x-2}}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{2}{\sqrt{x+h-2}} - \frac{2}{\sqrt{x-2}}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{2\sqrt{x-2} - 2\sqrt{x+h-2}}{\sqrt{x+h-2}\,\sqrt{x-2}}\right)$$

$$= \lim_{h\to 0} \frac{2}{h}\left(\frac{\sqrt{x-2} - \sqrt{x+h-2}}{\sqrt{x+h-2}\,\sqrt{x-2}}\right)\left(\frac{\sqrt{x-2}+\sqrt{x+h-2}}{\sqrt{x-2}+\sqrt{x+h-2}}\right)$$

$$= \lim_{h\to 0} \frac{2}{h}\left(\frac{(x-2)-(x+h-2)}{\sqrt{x+h-2}\,\sqrt{x-2}\,(\sqrt{x-2}+\sqrt{x+h-2})}\right)$$

$$= \lim_{h\to 0} \frac{2}{\cancel{h}} \cdot \frac{-\cancel{h}}{\sqrt{x+h-2}\,\sqrt{x-2}\,(\sqrt{x-2}+\sqrt{x+h-2})}$$

$$= \lim_{h\to 0} \frac{-2}{\sqrt{x+h-2}\,\sqrt{x-2}\,(\sqrt{x-2}+\sqrt{x+h-2})}$$

$$= \frac{-2}{\sqrt{x-2}\,\sqrt{x-2}\,(2\sqrt{x-2})} = \frac{-2}{2\,(x-2)^{3/2}}$$

$$= \frac{-1}{(x-2)^{3/2}}.$$

# Differentiation Rules

Q1 $f'(x) = 7x^6 + 12x^3 + 5e^x - 2x^{-2} + \frac{1}{2}x^{-\frac{1}{2}}.$

Q2 $g(t) = t^{5/2} + e^{\pi} + 9t^{-3} + \pi t^4$

$\Rightarrow g'(t) = \frac{5}{2}t^{3/2} + 0 - 27t^{-4} + 4\pi t^3.$

Q3 $y = x^3 e^x$

$\frac{dy}{dx} = (x^3)' e^x + x^3 (e^x)' = 3x^2 e^x + x^3 e^x.$

Q4 $f(x) = (x^2+4x+3)(5x^5 + x^{1/2})$

$f'(x) = (2x+4)(5x^5 + x^{1/2}) + (x^2+4x+3)(25x^4 + \frac{1}{2}x^{-\frac{1}{2}})$

Q5 $f(x) = \dfrac{3x^4+2}{x^2-2x+3}$

$f'(x) = \dfrac{(x^2-2x+3)(12x^3) - (3x^4+2)(2x-2)}{(x^2-2x+3)^2}$

Q6 $y = \dfrac{x^{2/3}}{\cos(x) + x^{5/7}}$

$\frac{dy}{dx} = \dfrac{(\cos(x) + x^{5/7})(\frac{2}{3}x^{-1/3}) - (x^{2/3})(-\sin(x) + \frac{5}{7}x^{-\frac{2}{7}})}{(\cos(x) + x^{5/7})^2}$

**Q7**

$$h(t) = (t^3+3t^2)^5$$

$$h'(t) = 5(t^3+3t^2)^4 \cdot (3t^2+6t)$$

**Q8**

$$g(t) = e^{t^3-4t^2+2t-5}$$

$$g'(t) = e^{t^3-4t^2+2t-5} \cdot (3t^2-8t+2)$$

**Q9**

$$y = (xe^x+5x^{10})(e^{3x})$$

$$\frac{dy}{dx} = (xe^x+5x^{10})'(e^{3x}) + (xe^x+5x^{10})(e^{3x})'$$

$$= (e^x + xe^x + 50x^9)e^{3x} + (xe^x+5x^{10})(3e^{3x})$$

**Q10**

$$h(x) = x^4 \sin(\cos(x))$$

$$h'(x) = 4x^3 \sin(\cos(x)) + x^4 \cdot \cos(\cos(x)) \cdot (-\sin(x$$

**Q11**

$$f(x) = (\sin(4x^2+3x))^5 \cdot \cos(3x+2)$$

$$f'(x) = 5(\sin(4x^2+3x))^4 \cdot \cos(4x^2+3x) \cdot (8x+3) \cdot \cos(3x+2$$

$$+ (\sin(4x^2+3x))^5 \cdot \sin(3x+2) \cdot 3.$$

Q12 $$f(x) = \left(e^{7x} + 5x^{9}\right)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}\left(e^{7x} + 5x^{9}\right)^{-\frac{1}{2}}\left(7e^{7x} + 45x^{8}\right)$$

Q13 $$f(x) = 3\left(x^{3} + 2x + 9\right)^{-5/3}$$

$$f'(x) = -\frac{5}{3} \cdot 3\left(x^{3} + 2x + 9\right)^{-8/3}\left(3x^{2} + 2\right)$$

Q14 $$f(x) = \tan\left(\frac{3x^{2} - 5}{e^{-x}}\right)$$

$$f'(x) = \sec^{2}\left(\frac{3x^{2} - 5}{e^{-x}}\right) \cdot \frac{e^{-x}(6x) - (3x^{2} - 5)(-e^{-x})}{(e^{-x})^{2}}$$

Q15 $$f(x) = \left(\frac{3x^{2} + 9x}{x^{3/2} - 7x^{5} + e^{2x}}\right)^{12}$$

$$f'(x) = 12\left(\frac{3x^{2} + 9x}{x^{3/2} - 7x^{5} + e^{2x}}\right)^{11} \cdot \left(\frac{(x^{3/2} - 7x^{5} + e^{2x})(6x + 9) - (3x^{2} + 9x)\left(\frac{3}{2}x^{\frac{1}{2}} - 35x^{4} + 2e^{2x}\right)}{(x^{3/2} - 7x^{5} + e^{2x})^{2}}\right)$$

Q16 $f(x) = \sin(\cos(\tan(x)))$

$$f'(x) = \cos(\cos(\tan(x))) \cdot (-\sin(\tan(x))) \cdot \sec^2(x)$$

Q17 $f(x) = e^{\sin(3x)}$

$$f'(x) = e^{\sin(3x)} \cdot \cos(3x) \cdot 3$$

Q18 $f(x) = \dfrac{x^3 (4x^2-3)^{1/2}}{x^{2/9} + 4x^{12}}$

$$f'(x) = \frac{(x^{2/9}+4x^{12})\left(3x^2(4x^2-3)^{1/2} + x^3 \cdot \frac{1}{2}(4x^2-3)^{-\frac{1}{2}}(8x)\right) - \left(\frac{2}{9}x^{-\frac{7}{9}} + 48x^{11}\right)\left(x^3(4x^2-3)^{1/2}\right)}{(x^{2/9} + 4x^{12})^2}$$