Definition of the Derivative

Recall the formula for finding the slope of a line joining two points \((x_1, y_1)\) and \((x_2, y_2)\):

\[
m = \frac{y_2 - y_1}{x_2 - x_1}.
\]

Suppose these two points are points on a curve \(y = f(x)\):

Note that \(y_1 = f(x_1)\) and \(y_2 = f(x_2)\). Let’s change our notation slightly: instead of \(x_1\) and \(x_2\), we use \(x\) and \(x + h\). Also, instead of \(y_1\) and \(y_2\), we use \(f(x)\) and \(f(x + h)\) as shown below.
Notice that the formula for the slope of the line joining the two points has now become

\[
\frac{f(x + h) - f(x)}{x + h - x} = \frac{f(x + h) - f(x)}{h}
\]

Now, suppose we move those two points closer together (we do this by taking a smaller number for \(h\)):

![Graph showing the slope of a tangent line]

The closer we push the second point towards \((x, f(x))\), the closer the above formula comes to giving us the slope of the tangent line to the curve at that point.

The expression is not defined for \(h = 0\), but this is where limits come in handy – we can approximate the slope of the tangent line by taking the limit of the above expression as \(h\) approaches 0 (if this limit exists). We call this the derivative of the function \(f(x)\), and denote it \(f'(x)\) or \(\frac{d}{dx} (f(x))\).

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

We now have a function describing the slope of the tangent line to the curve!

**Example 1.** Using the definition of the derivative, find \(f'(x)\) for the function

\[
f(x) = 2x^2
\]

\[
f(x) = 2x^2 \quad \Rightarrow \quad f(x + h) = 2(x + h)^2
\]
\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]
\[ = \lim_{h \to 0} \frac{2(x + h)^2 - 2x^2}{h} \]
\[ = \lim_{h \to 0} \frac{2(x^2 + 2xh + h^2) - 2x^2}{h} \]
\[ = \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - 2x^2}{h} \]
\[ = \lim_{h \to 0} \frac{4xh + 2h^2}{h} \]
\[ = \lim_{h \to 0} \frac{h(4x + 2h)}{h} \]
\[ = \lim_{h \to 0} 4x + 2h \]
\[ = 4x + 2(0) = 4x \]

**Example 2.** Using the definition of the derivative, find \( f'(x) \) for the function

\[ f(x) = \sqrt{x + 2} \]

**Solution:**

\[ f(x) = \sqrt{x + 2} \implies f(x + h) = \sqrt{x + h + 2} \]

\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]
\[ = \lim_{h \to 0} \frac{\sqrt{x + h + 2} - \sqrt{x + 2}}{h} \]
\[ = \lim_{h \to 0} \frac{\sqrt{x + h + 2} - \sqrt{x + 2}}{h} \cdot \frac{\sqrt{x + h + 2} + \sqrt{x + 2}}{\sqrt{x + h + 2} + \sqrt{x + 2}} \]
\[ = \lim_{h \to 0} \frac{x + h + 2 - (x + 2)}{h(\sqrt{x + h + 2} + \sqrt{x + 2})} \]
\[ = \lim_{h \to 0} \frac{h}{h(\sqrt{x + h + 2} + \sqrt{x + 2})} \]
\[ = \lim_{h \to 0} \frac{1}{\sqrt{x + h + 2} + \sqrt{x + 2}} \]
\[ = \frac{1}{\sqrt{x + 0 + 2} + \sqrt{x + 2}} \]
\[ = \frac{1}{2\sqrt{x + 2}} \]

**Example 3.** Using the definition of the derivative, find \( f'(x) \) for the function

\[ f(x) = \frac{1}{x} \]
Solution:

\[ f(x) = \frac{1}{x} \implies f(x + h) = \frac{1}{x + h} \]

\[
\begin{align*}
    f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
    &= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\
    &= \lim_{h \to 0} \frac{1}{h} \cdot \frac{x - (x+h)}{x(x+h)} \\
    &= \lim_{h \to 0} \frac{x - x - h}{h} \cdot \frac{x}{x(x+h)} \\
    &= \lim_{h \to 0} \frac{-h}{h} \cdot \frac{x}{x(x+h)} \\
    &= \lim_{h \to 0} -1 \cdot \frac{1}{x} \\
    &= -\frac{1}{x(x + 0)} \\
    &= -\frac{1}{x^2}
\end{align*}
\]

Exercises

For each of the following functions, find \( f'(x) \) using the definition of the derivative.

1. \( f(x) = x^2 + 6x - 3 \)
2. \( f(x) = \sqrt{x} - 4 \)
3. \( f(x) = \frac{2}{x^2} \)
4. \( f(x) = \frac{1}{\sqrt{x} + 1} \)