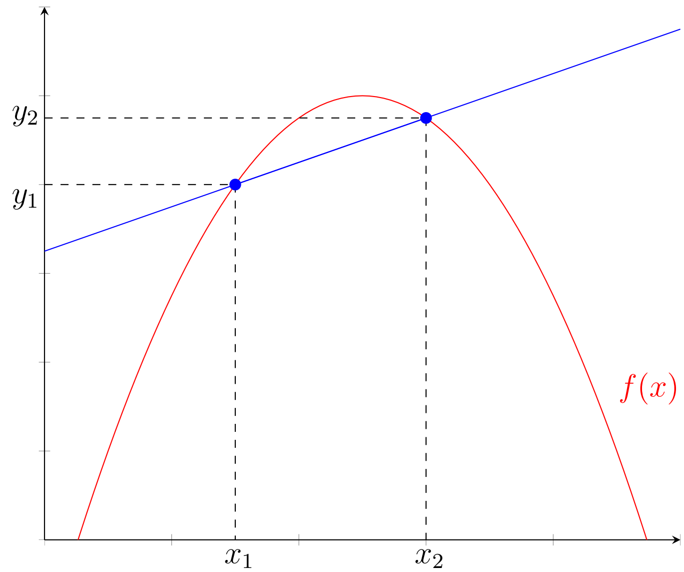
Definition of the Derivative

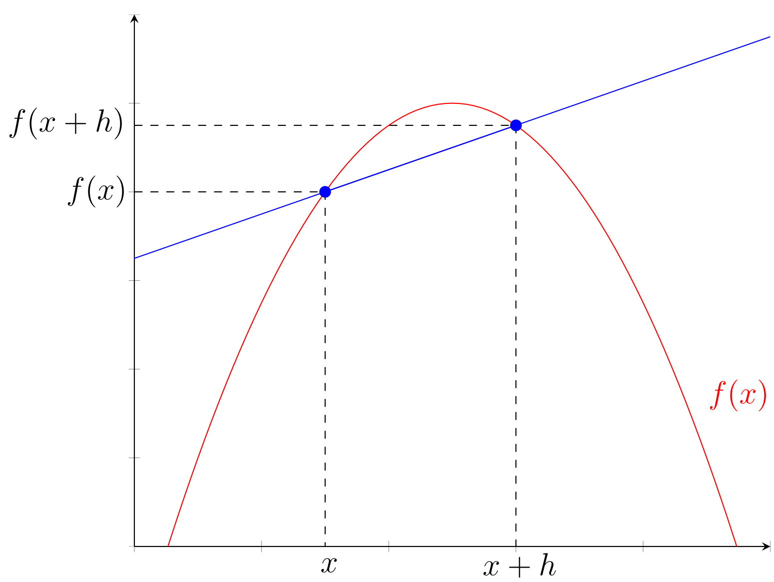
Recall the formula for finding the slope of a line joining two points (x_1, y_1) and (x_2, y_2):

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Suppose these two points are points on a curve $y = f(x)$:



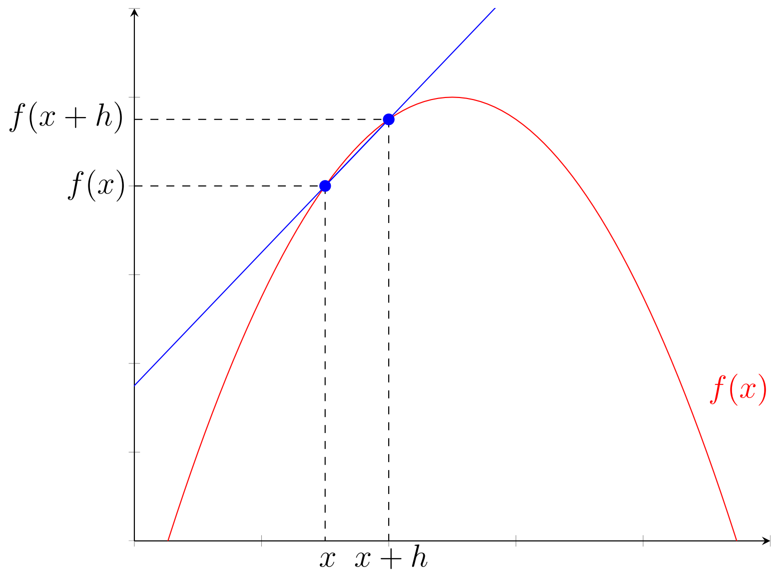
Note that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Let's change our notation slightly: instead of x_1 and x_2, we use x and $x + h$. Also, instead of y_1 and y_2, we use $f(x)$ and $f(x + h)$ as shown below.



Notice that the formula for the slope of the line joining the two points has now become

$$\frac{f(x+h)-f(x)}{x+h-x} = \frac{f(x+h)-f(x)}{h}$$

Now, suppose we move those two points closer together (we do this by taking a smaller number for h):



The closer we push the second point towards $(x, f(x))$, the closer the above formula comes to giving us the slope of the tangent line to the curve at that point.

The expression is not defined for $h = 0$, but this is where limits come in handy – we can approximate the slope of the tangent line by taking the limit of the above expression as h approaches 0 (if this limit exists). We call this the derivative of the function $f(x)$, and denote it $f'(x)$ or $\frac{d}{dx}(f(x))$.

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}}$$

We now have a function describing the slope of the tangent line to the curve!

Example 1. Using the definition of the derivative, find $f'(x)$ for the function

$$f(x) = 2x^2$$

$$f(x) = 2x^2 \quad \Longrightarrow \quad f(x+h) = 2(x+h)^2$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{2(x+h)^2-2x^2}{h} \\
&= \lim_{h\to 0} \frac{2(x^2+2xh+h^2)-2x^2}{h} \\
&= \lim_{h\to 0} \frac{2x^2+4xh+2h^2-2x^2}{h} \\
&= \lim_{h\to 0} \frac{4xh+2h^2}{h} \\
&= \lim_{h\to 0} \frac{h(4x+2h)}{h} \\
&= \lim_{h\to 0} 4x+2h \\
&= 4x+2(0) = 4x
\end{aligned}$$

Example 2. Using the definition of the derivative, find $f'(x)$ for the function

$$f(x) = \sqrt{x+2}$$

Solution:

$$f(x) = \sqrt{x+2} \quad \Longrightarrow \quad f(x+h) = \sqrt{x+h+2}$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h+2}-\sqrt{x+2}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h+2}-\sqrt{x+2}}{h} \cdot \frac{\sqrt{x+h+2}+\sqrt{x+2}}{\sqrt{x+h+2}+\sqrt{x+2}} \\
&= \lim_{h\to 0} \frac{x+h+2-(x+2)}{h(\sqrt{x+h+2}+\sqrt{x+2})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+2}+\sqrt{x+2})} \\
&= \lim_{h\to 0} \frac{1}{(\sqrt{x+h+2}+\sqrt{x+2})} \\
&= \frac{1}{\sqrt{x+0+2}+\sqrt{x+2}} \\
&= \frac{1}{2\sqrt{x+2}}
\end{aligned}$$

Example 3. Using the definition of the derivative, find $f'(x)$ for the function

$$f(x) = \frac{1}{x}$$

Solution:

$$f(x) = \frac{1}{x} \quad \Longrightarrow \quad f(x+h) = \frac{1}{x+h}$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{x+h}\cdot\frac{x}{x} - \frac{1}{x}\cdot\frac{x+h}{x+h}}{h} \\
&= \lim_{h\to 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{x-x-h}{x(x+h)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{-h}{x(x+h)}}{h} \\
&= \lim_{h\to 0} \frac{-h}{x(x+h)}\cdot\frac{1}{h} \\
&= \lim_{h\to 0} \frac{-\cancel{h}}{x(x+h)}\cdot\frac{1}{\cancel{h}} \\
&= \lim_{h\to 0} \frac{-1}{x(x+h)} \\
&= \frac{-1}{x(x+0)} \\
&= -\frac{1}{x^2}
\end{aligned}$$

Exercises

For each of the following functions, find $f'(x)$ using the definition of the derivative.

1. $f(x) = x^2 + 6x - 3$
2. $f(x) = \sqrt{x-4}$
3. $f(x) = \dfrac{2}{x^2}$
4. $f(x) = \dfrac{1}{\sqrt{x+1}}$