De Moivre's Theorem

Theorem. *Let* $z = r(\cos(\theta) + i\sin(\theta))$ *be a complex number. Then for any positive integer* n*,*

$$\boxed{z^n = r^n(\cos(n\theta) + i\sin(n\theta))}$$

In Math 1210, De Moivre's theorem is used in two ways:

1. To aid in simplifying complex number expressions.
2. To find roots of complex numbers.
3. To prove some trig identities.

Let's start with simplifying complex expressions.

Example. *Simplify the expression*

$$(-2\sqrt{3} - 2i)^{12}$$

Solution:
Without De Moivre's theorem to help, we would be faced with the unenviable task of the following multiplication:

$$(-2\sqrt{3} - 2i)(-2\sqrt{3} - 2i)(-2\sqrt{3} - 2i)(-2\sqrt{3} - 2i)(-2\sqrt{3} - 2i)....(-2\sqrt{3} - 2i)$$

Thankfully, we do not need to do this. To use De Moivre's theorem, we first need write the expression in the form $r(\cos(\theta) + i\sin(\theta))$.

- $r = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = \sqrt{4(3) + 4} = \sqrt{16} = 4.$
- $\theta = \tan^{-1}\left(\dfrac{-2}{-2\sqrt{3}}\right) = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6}$ or $\dfrac{7\pi}{6}$. The answer is $\dfrac{7\pi}{6}$ since $-2\sqrt{3} - 2i$ is in the third quadrant.

Therefore,

$$-2\sqrt{3} - 2i = 4\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right)$$

Now we can apply De Moivre's theorem!

$$\begin{aligned}
(-2\sqrt{3} - 2i)^{12} &= \left(4\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right)\right)^{12} \\
&= 4^{12}\left(\cos\left(12 \cdot \frac{7\pi}{6}\right) + i\sin\left(12 \cdot \frac{7\pi}{6}\right)\right) \\
&= 4^{12}\left(\cos(14\pi) + i\sin(14\pi)\right) \\
&= 4^{12}\left(\cos(0) + i\sin(0)\right) \\
&= 4^{12}\left(1 + 0i\right) \\
&= 4^{12}
\end{aligned}$$

De Moivre's theorem can also be used to find roots of complex numbers!

Example. *Find all solutions to the equation*

$$z^4 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Solution:

- $r = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1.$
- $\theta = \tan^{-1}\left(\frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right) = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$ or $\frac{4\pi}{3}$. The answer is $\frac{4\pi}{3}$ since $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$ is in the third quadrant. Therefore,

$$-\frac{1}{2} - i\frac{\sqrt{3}}{2} = 1\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right)$$

Let $z = r(\cos\theta + i\sin\theta)$.

$$\text{Set } z^4 = r^4\left(\cos(4\theta) + i\sin(4\theta)\right) = 1\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right).$$

$$\begin{aligned} \implies r &= 1, \\ 4\theta &= \frac{4\pi}{3} + 2\pi k \\ \implies \theta &= \frac{\pi}{3} + \frac{\pi k}{2} \end{aligned}$$

For $k = 1, 2, 3, 4$,

$$\begin{aligned} z_0 &= \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2} \\ z_1 &= \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} + i\frac{1}{2} \\ z_2 &= \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2} \\ z_3 &= \cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right) = \frac{\sqrt{3}}{2} - i\frac{1}{2} \end{aligned}$$

Finally, let's see how De Moivre's theorem can be used in proving a trig identity.

Example. *Use De Moivre's theorem to prove*

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta.$$

Solution:

By De Moivre's theorem,

$$(\cos(\theta) + i\sin(\theta))^3 = \cos(3\theta) + i\sin(3\theta) \tag{1}$$

Let's briefly focus on the left side of the above equation. Multiplying everything out (or using the binomial theorem if you know it),

$$\begin{aligned}(\cos(\theta) + i\sin(\theta))^3 &= \cos^3(\theta) + 3i\cos^2(\theta)\sin(\theta) + 3i^2\cos(\theta)\sin^2(\theta) + i^3\sin^3(\theta)\\ &= \cos^3(\theta) + 3i\cos^2(\theta)\sin(\theta) - 3\cos(\theta)\sin^2(\theta) - i\sin^3(\theta)\\ &= (\cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)) + i(3\cos^2(\theta)\sin(\theta) - \sin^3(\theta))\end{aligned}$$

Subbing this back into (1),

$$(\cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)) + i(3\cos^2(\theta)\sin(\theta) - \sin^3(\theta)) = \cos(3\theta) + i\sin(3\theta)$$

Now, if we equate the real part on the left side of the equation with the real part on the right, we get the result:

$$\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)$$

Exercises

1. Calculate $(-4+4i)^4$.
2. Find all solutions of the equation $z^5 = -1$.
3. Using De Moivre's theorem, show that

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta).$$