Math Help Centre

Curve Sketching

For each of the following functions, their derivatives and second derivatives

1. $f(x) = \dfrac{2+x}{(2-x)^2}$, $\quad f'(x) = \dfrac{x+6}{(2-x)^3}$, $\quad f''(x) = \dfrac{2(x+10)}{(2-x)^4}$ (Fall 2013)

2. $g(x) = \dfrac{(x+1)^2}{x^2+1}$, $\quad g'(x) = \dfrac{2(1-x^2)}{(x^2+1)^2}$, $\quad g''(x) = \dfrac{4x(x^2-3)}{(x^2+1)^3}$ (Winter 2014)

3. $h(x) = \dfrac{x}{x^3-1}$, $\quad h'(x) = -\dfrac{1+2x^3}{(x^3-1)^2}$, $\quad h''(x) = \dfrac{6x^2(x^3+2)}{(x^3-1)^3}$ (Fall 2016)

find

- Domain
- Symmetry (is the function even/odd/neither?)
- Coordinate(s) of x-intercepts and y-intercepts.
- Equation(s) of horizontal asymptotes (calculate all limits associated with those asymptotes).
- Equation(s) of vertical asymptotes (calculate all limits associated with those asymptotes).
- Intervals of increase/decrease.
- Coordinates of local maxima and minima.
- Intervals of concave up/down.
- Coordinates of inflection points.
- Sketch the graph, labelling all special points and asymptotes found above.

Curve Sketching Solutions

1. Domain : $(-\infty, 2) \cup (2, \infty)$

Symmetry : $f(-x) = \dfrac{2+(-x)}{(2-(-x))^2} = \dfrac{2-x}{(2+x)^2} \neq f(x)$
$\neq -f(x)$

No symmetry → Neither even nor odd

x-int's : Set $y=0$. $\dfrac{2+x}{(2-x)^2} = 0 \Rightarrow x = -2$.

x-int at $(-2, 0)$.

y-int : Set $x=0$ $y = \dfrac{2+0}{(2-0)^2} = \dfrac{1}{2}$

y-int at $(0, \frac{1}{2})$.

H.A.'s :

$$\lim_{x\to\infty} \frac{2+x}{(2-x)^2} = \lim_{x\to\infty} \frac{2+x}{4-4x+x^2} = \lim_{x\to\infty} \frac{\frac{2}{x^2}+\frac{x}{x^2}}{\frac{4}{x^2}-\frac{4x}{x^2}+\frac{x^2}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{2}{x^2}+\frac{1}{x}}{\frac{4}{x^2}-\frac{4}{x}+1} = \frac{0+0}{0-0+1} = \frac{0}{1} = 0.$$

Likewise, $\displaystyle\lim_{x\to-\infty} \frac{2+x}{(2-x)^2} = 0$

$\Rightarrow$ H.A. : $\boxed{y=0}$

$$\lim_{x\to 2^-} \frac{2+x}{(2-x)^2} = +\infty .$$

V.A. : $\boxed{x = 2}$

$$\lim_{x\to 2^+} \frac{2+x}{(2-x)^2} = +\infty .$$

Intervals of increase/decrease :

$f'(x) = \dfrac{x+6}{(2-x)^3}$. Critical number : $x=-6$.

	$(-\infty,-6)$	$(-6,2)$	$(2,\infty)$
$x+6$	$-$	$+$	$+$
$(2-x)^3$	$+$	$+$	$-$
	$-$	$+$	$-$

Interval of increase : $(-6,2)$

Intervals of decrease : $(-\infty,-6)\cup(2,\infty)$

Local maxima/minima :

Critical number : $x=-6$.

$$f(-6) = \frac{2-6}{(2-(-6))^2} = \frac{-4}{8^2} = -\frac{4}{64} = -\frac{1}{16}$$

coordinate of critical number : $(-6,-\frac{1}{16})$

From table : $\searrow (-6,-\frac{1}{16}) \nearrow$

$\Rightarrow (-6,-\frac{1}{16})$ is a local min.

Concave up/down:

$$f''(x) = \frac{2(x+10)}{(2-x)^4}.$$

Inflection at $x=-10$.

	$(-\infty,-10)$	$(-10,2)$	$(2,\infty)$
$x+10$	$-$	$+$	$+$
$(2-x)^4$	$+$	$+$	$+$
	$-$	$+$	$+$

Concave up: $(-10,2) \cup (2,\infty)$

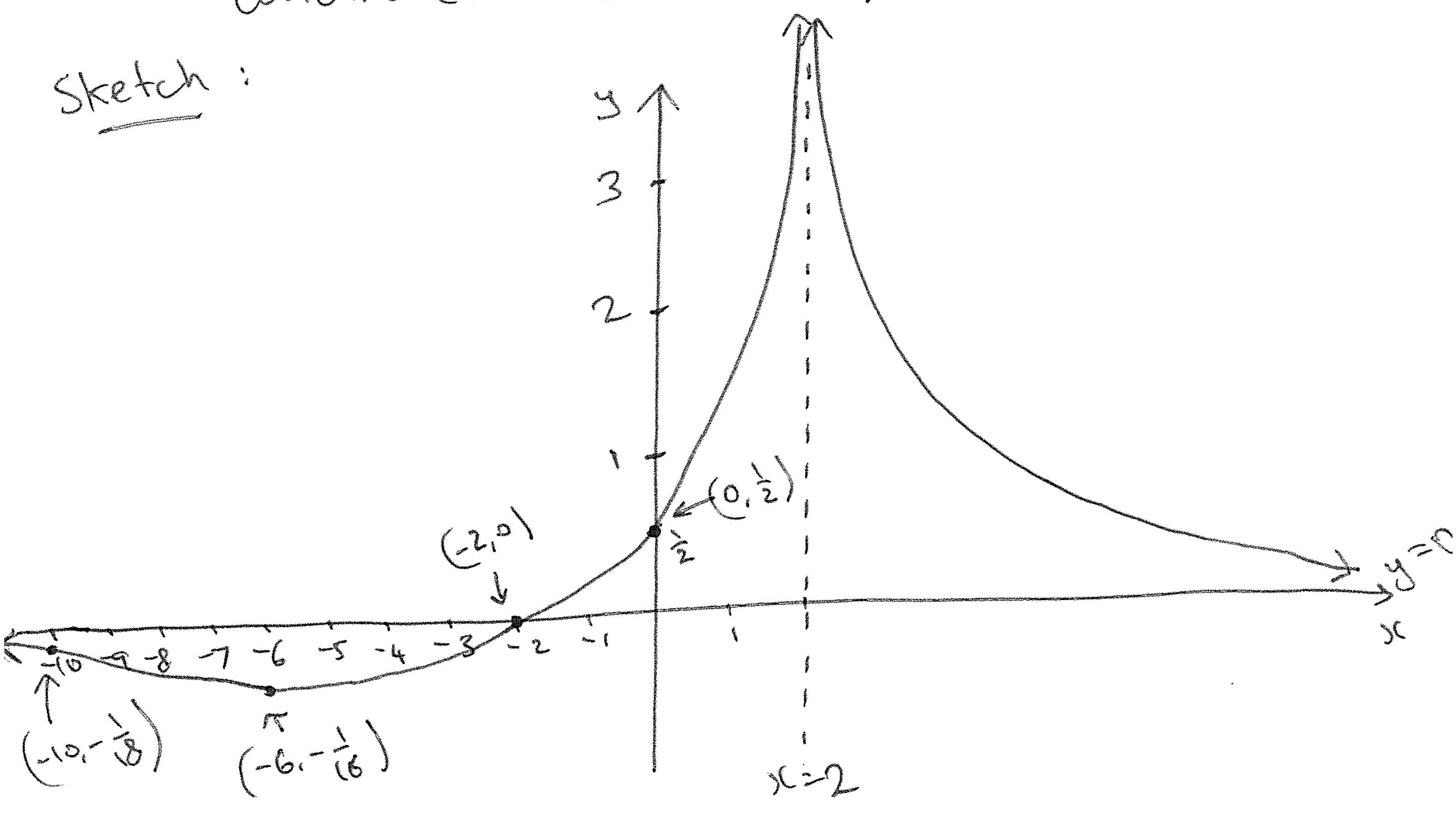
Concave down: $(-\infty,-10)$.

Inflection point:

$$f(-10) = \frac{2-10}{(2-(-10))^2} = \frac{-8}{(12)^2} = \frac{-8}{144} = -\frac{1}{18}$$

coordinates: $(-10, -\frac{1}{18})$

Sketch:



2. Domain : $(-\infty, \infty)$

Symmetry : $g(-x) = \dfrac{(-x+1)^2}{(-x)^2+1} = \dfrac{(-x+1)^2}{x^2+1} \neq g(x)$
$\neq -g(x)$

$\Rightarrow$ No symmetry.

x-int's : Set $y=0$ $\quad \dfrac{(x+1)^2}{x^2+1} = 0 \Rightarrow (x+1)^2 = 0$
$\Rightarrow x = -1.$

x-int at $(-1, 0)$

y-int : Set $x=0$. $\Rightarrow y = \dfrac{(0+1)^2}{(0)^2+1} = \dfrac{1}{1} = 1$

y-int at $(0, 1)$.

H.A.'s :

$$\lim_{x\to\infty} \frac{(x+1)^2}{x^2+1} = \lim_{x\to\infty} \frac{x^2+2x+1}{x^2+1} = \lim_{x\to\infty} \frac{\frac{x^2}{x^2} + \frac{2x}{x^2} + \frac{1}{x^2}}{\frac{x^2}{x^2} + \frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{1 + \frac{2}{x} + \frac{1}{x^2}}{1 + \frac{1}{x^2}} = \frac{1+0+0}{1+0} = 1$$

Likewise, $\lim_{x\to-\infty} g(x) = 1$

$\Rightarrow$ H.A. at $\boxed{y=1}$

V.A.'s : None.

Intervals of increase/decrease

$$g'(x) = \frac{2(1-x^2)}{(x^2+1)^2} = \frac{2(1-x)(1+x)}{(x^2+1)^2}.$$

Critical numbers at $x=1$, $x=-1$.

	$(-\infty,-1)$	$(-1,1)$	$(1,\infty)$
$2(1-x)$	+	+	−
$1+x$	−	+	+
$(x^2+1)^2$	+	+	+
	−	+	−

Increasing on $(-1,1)$

Decreasing on $(-\infty,-1) \cup (1,\infty)$.

Local maxima/minima :

Critical number $x=1$: $g(1) = \dfrac{(1+1)^2}{(1)^2+1} = \dfrac{4}{2} = 2$

$(1,2)$

From table $\nearrow (1,2) \searrow$ $\Rightarrow$ $(1,2)$ is local max.

Critical number $x=-1$: $g(-1) = \dfrac{(-1+1)^2}{(-1)^2+1} = \dfrac{0}{2} = 0$

$(-1,0)$

From table $\searrow (-1,0) \nearrow$ $\Rightarrow$ $(-1,0)$ is local min

Intervals of concave up/down

$$g''(x) = \frac{4x(x^2-3)}{(x^2+1)^3} = \frac{4x(x+\sqrt{3})(x-\sqrt{3})}{(x^2+1)^3}.$$

Inflection numbers at $x=0$, $x=-\sqrt{3}$, $x=+\sqrt{3}$.

	$(-\infty,-\sqrt{3})$	$(-\sqrt{3},0)$	$(0,\sqrt{3})$	$(\sqrt{3},\infty)$
$4x$	$-$	$-$	$+$	$+$
$x+\sqrt{3}$	$-$	$+$	$+$	$+$
$x-\sqrt{3}$	$-$	$-$	$-$	$+$
$(x^2+1)^3$	$+$	$+$	$+$	$+$
	$-$	$+$	$-$	$+$

Inflection points:

$x=-\sqrt{3}$: $g(-\sqrt{3}) = \dfrac{(-\sqrt{3}+1)^2}{(-\sqrt{3})^2+1} = \dfrac{3-2\sqrt{3}+1}{3+1} = \dfrac{4-2\sqrt{3}}{4}$

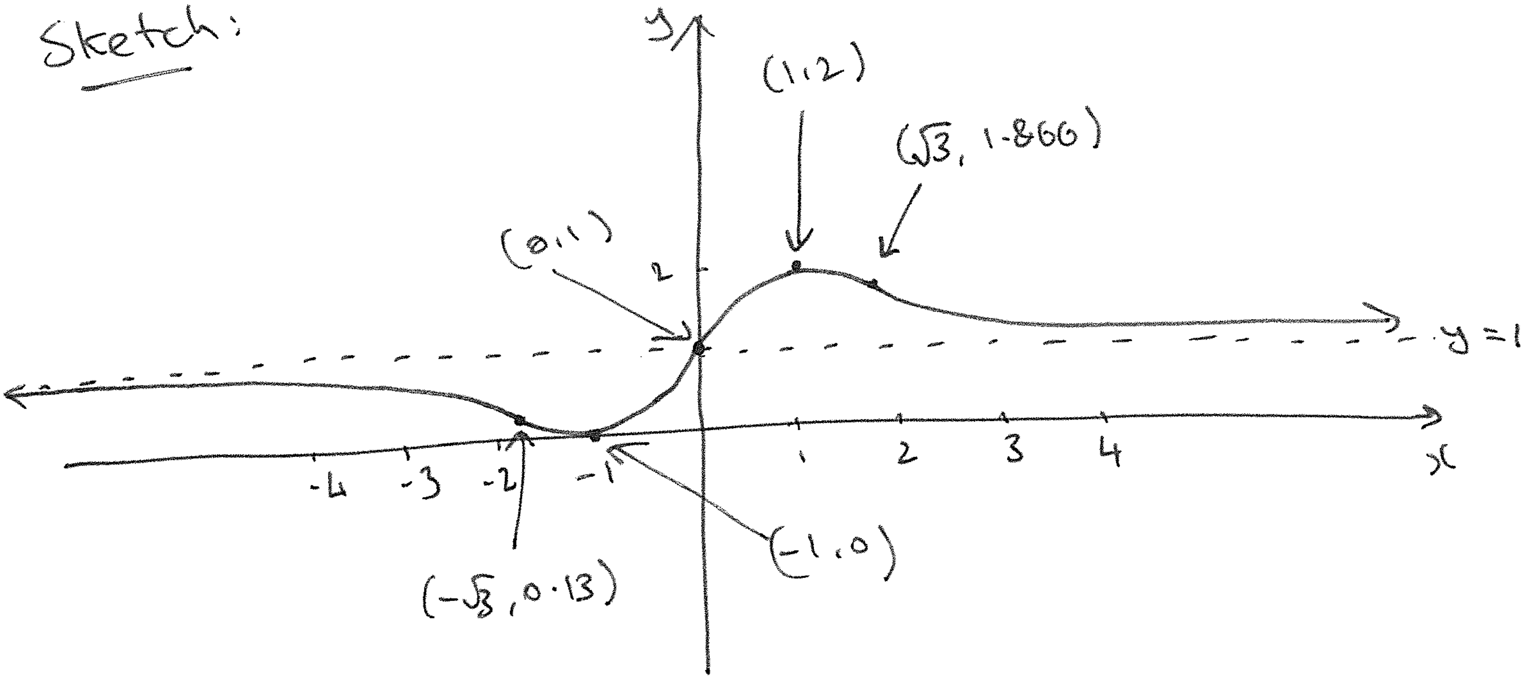
$= 1-\dfrac{\sqrt{3}}{2} \approx 0.13$ $\quad\boxed{(-\sqrt{3},\ 0.13)}$

$x=+\sqrt{3}$: $g(\sqrt{3}) = \dfrac{(\sqrt{3}+1)^2}{(\sqrt{3})^2+1} = \dfrac{3+2\sqrt{3}+1}{3+1} = \dfrac{4+2\sqrt{3}}{4}$

$= 1+\dfrac{\sqrt{3}}{2} \approx 1.866$ $\quad\boxed{(\sqrt{3},\ 1.866)}$

$x=0$: $g(0) = \dfrac{(0+1)^2}{(0)^2+1} = \dfrac{1}{1} = 1$ $\quad\boxed{(0,1)}$

Sketch:



3. Domain : $\dfrac{x}{x^3-1} = \dfrac{x}{(x-1)\underbrace{(x^2+x+1)}_{\text{Never zero.}}}$

$(-\infty, 1) \cup (1, \infty)$

Symmetry : $h(-x) = \dfrac{-x}{(-x)^3-1} = \dfrac{-x}{-x^3-1} = \dfrac{x}{x^3+1}$ $\neq h(x)$ $\neq -h(x)$

No symmetry.

x-int's : Set $y=0$.

$\dfrac{x}{x^3-1} = 0 \Rightarrow x = 0$ $(0,0)$

y-int : Set $x=0$

$\dfrac{0}{(0)^3-1} = \dfrac{0}{-1} = 0$ $(0,0)$

H. A.'s :

$$\lim_{x\to\infty} \frac{x}{x^3-1} = \lim_{x\to\infty} \frac{\frac{x}{x^3}}{\frac{x^3}{x^3} - \frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{1}{x^2}}{1-\frac{1}{x^3}} = \frac{0}{1-0} = 0$$

Likewise, $\lim_{x\to-\infty} \frac{x}{x^3-1} = 0$. H. A. : $\boxed{y=0}$

V. A.'s : $x = 1$

$$\lim_{x\to 1^-} \frac{x}{x^3-1} = \lim_{x\to 1^-} \frac{\overset{+}{x}}{\underset{-}{(x-1)}\underset{+}{(x^2+x+1)}} = -\infty.$$

$$\lim_{x\to 1^+} \frac{\overset{+}{x}}{\underset{+}{(x-1)}\underset{+}{(x^2+x+1)}} = +\infty.$$

Intervals of increase/decrease :

$$h'(x) = \frac{-(1+2x^3)}{(x^3-1)^2}.$$

Critical number where

$1+2x^3 = 0$

$\Rightarrow x^3 = -\frac{1}{2} \Rightarrow x = \left(-\frac{1}{2}\right)^{\frac{1}{3}}$

	$\left(-\infty, \left(-\frac{1}{2}\right)^{\frac{1}{3}}\right)$	$\left(\left(-\frac{1}{2}\right)^{3}, 1\right)$	$(1, \infty)$
$-(1+2x^3)$	+	−	−
$(x^3-1)^2$	+	+	+
	+	−	+

Increasing on $\left(-\infty, \left(-\frac{1}{2}\right)^{\frac{1}{3}}\right) \cup (1, \infty)$

Decreasing on $\left(\left(-\frac{1}{2}\right)^{3}, 1\right)$

Local max/min :

Critical number at $\left(-\frac{1}{2}\right)^{\frac{1}{3}}$

$$h\left(\left(-\frac{1}{2}\right)^{3}\right) = \frac{\left(-\frac{1}{2}\right)^{\frac{1}{3}}}{\left(\left(-\frac{1}{2}\right)^{\frac{1}{3}}\right)^{3} - 1} = \frac{\left(-\frac{1}{2}\right)^{\frac{1}{3}}}{-\frac{1}{2} - 1} = -\frac{2}{3}\left(-\frac{1}{2}\right)^{\frac{1}{3}}$$

$\Rightarrow$ Critical point: $\left(\left(-\frac{1}{2}\right)^{\frac{1}{3}}, -\frac{2}{3}\left(-\frac{1}{2}\right)^{\frac{1}{3}}\right)$

From table : $\nearrow \left(\left(-\frac{1}{2}\right)^{\frac{1}{3}}, -\frac{2}{3}\left(-\frac{1}{2}\right)^{\frac{1}{3}}\right) \searrow$

$\Rightarrow$ Local max $\nearrow$.

Concave up/down :

$$h''(x) = \frac{6x^2(x^3+2)}{(x^3-1)^3}$$

Inflection numbers when

$* \ 6x^2 = 0 \Rightarrow x = 0$

$* \ x^3 + 2 = 0 \Rightarrow x = (-2)^{\frac{1}{3}}$

	$(-\infty, -2^{\frac{1}{3}})$	$(-2^{\frac{1}{3}}, 0)$	$(0, 1)$	$(1, \infty)$
$6x^2$	+	+	+	+
x^3+2	−	+	+	+
$(x^3-1)^3$	−	−	−	+
	+	−	−	+

Inflection points

$x = 0 \quad : \quad h(0) = \dfrac{0}{0^3 - 1} = 0 \qquad (0, 0)$

$x = -2^{\frac{1}{3}} \quad : \quad h(-2^{\frac{1}{3}}) = \dfrac{-2^{\frac{1}{3}}}{(-2^{\frac{1}{3}})^3 - 1} = \dfrac{-2^{\frac{1}{3}}}{-2-1} = -\frac{1}{3}(-2)^{\frac{1}{3}}$

$\left(-2^{\frac{1}{3}}, -\frac{1}{3}(-2)^{\frac{1}{3}}\right)$

Sketch: