Abstract. Let $L_1$ be a finite lattice with an ideal $L_2$. Then the restriction map is a $\{0, 1\}$-homomorphism from $\text{Con} \ L_1$ into $\text{Con} \ L_2$.

In 1986, the present authors published the converse. If $D_1$ and $D_2$ are finite distributive lattices, and $\varphi : D_1 \to D_2$ is a $\{0, 1\}$-homomorphism, then there are finite lattices $L_1$ and $L_2$ with an embedding $\eta$ of $L_2$ as an ideal of $L_1$, and there are isomorphisms $\varepsilon_1 : \text{Con} \ L_1 \to D_1$ and $\varepsilon_2 : \text{Con} \ L_2 \to D_2$ such that $\varphi$ is represented as the restriction map of congruences from $L_1$ to $L_2$, up to the two isomorphisms.

Let us call a lattice isoform, if for any congruence, all congruence classes are isomorphic lattices. In 2003, G. Grätzer and E. T. Schmidt proved that every finite distributive lattice can be represented as the congruence lattice of an isoform lattice.

In this paper we combine the two results, reproving the 1986 result with isoform lattices.

1. Introduction

Let $\eta : K_2 \to K_1$ be a homomorphism of lattices. There is then a $\{1, \land\}$-preserving mapping $\text{res}(\eta) : \text{Con} \ K_1 \to \text{Con} \ K_2$, the restriction map, defined by setting $a \equiv b \ (\text{res}(\eta) \Theta)$ iff $\eta a \equiv \eta b \ (\Theta)$. The mapping $\text{res}(\eta)$ is 0-preserving exactly when $\eta$ is an embedding. If $\text{res}(\eta)$ is a bijection, and so is a lattice isomorphism, then we say that $\eta$ is a congruence-preserving extension.

Now, in general, even if $\eta$ is an embedding, $\text{res}(\eta)$ need not preserve joins. However, if $\eta$ embeds $K_2$ as a convex sublattice of $K_1$, in particular, as an ideal of $K_1$, then $\text{res}(\eta)$ also preserves joins, and so is a $\{0, 1\}$-preserving lattice homomorphism.

A congruence relation $\Theta$ on a lattice $L$ is said to be isoform, if all the congruence classes of $\Theta$ are isomorphic sublattices of $L$. The lattice $L$ is said to be isoform, if all of its congruences are isoform.

In this paper we prove the following theorem:

**Theorem 1.** Let $K_1$ and $K_2$ be lattices with finite congruence lattices, $\text{Con} \ K_1$ and $\text{Con} \ K_2$. Let $\varphi : \text{Con} \ K_1 \to \text{Con} \ K_2$ be a $\{0, 1\}$-preserving lattice homomorphism. Then there are isoform lattices $L_1$ and $L_2$ with an embedding $\eta$ of $L_2$ as an ideal in $L_1$, and there are congruence-preserving extensions $\varepsilon_1 : K_1 \to L_1$ and $\varepsilon_2 : K_2 \to L_2$ such that

$$\text{res}(\varepsilon_2) \text{res}(\eta) \Theta = \varphi \text{res}(\varepsilon_1) \Theta.$$
for all \( \Theta \in \text{Con} L_1 \). If \( K_2 \) is finite, then \( L_2 \) can be taken to be finite. If \( K_1 \) is also finite, then \( L_1 \) can also be taken to be finite.

By a classical result, which in [1] we call the Dilworth Theorem, every finite distributive lattice can be represented as the congruence lattice of a finite lattice. Combining the Dilworth Theorem with Theorem 1, we then obtain:

**Theorem 2.** Let \( D_1 \) and \( D_2 \) be finite distributive lattices, and let \( \varphi: D_1 \to D_2 \) be a \( \{0,1\} \)-preserving lattice homomorphism. Then there are finite isoform lattices \( L_1 \) and \( L_2 \) with an embedding \( \eta \) of \( L_2 \) as an ideal of \( L_1 \), and there are isomorphisms \( \varepsilon_1: \text{Con} L_1 \to D_1 \) and \( \varepsilon_2: \text{Con} L_2 \to D_2 \) such that

\[
\varepsilon_2 \res(\eta) \Theta = \varphi \varepsilon_1 \Theta,
\]

for all \( \Theta \in \text{Con} L_1 \).

In the process of proving Theorem 1, we shall develop all the tools needed to prove Theorem 2 without needing to appeal to the Dilworth Theorem.

Some of the techniques we utilize were developed in G. Grätzer and E. T. Schmidt [4] and G. Grätzer, R. W. Quackenbush, and E. T. Schmidt [5]. An aspect of this work, the solution of Problem 6 of G. Grätzer, R. W. Quackenbush, and E. T. Schmidt [5], was written up as a separate paper: G. Grätzer, H. Lakser, and R. W. Quackenbush [3]. In view of the result in this paper, the following problem arises naturally:

**Problem.** Let \( K_1 \) and \( K_2 \) be lattices with finite congruence lattices, let \( K_2 \) be an ideal of \( K_1 \). Do there exist isoform lattices \( L_1 \) and \( L_2 \) with \( L_2 \) an ideal in \( L_1 \), such that \( L_1 \) is a congruence-preserving extension of \( K_1 \) and \( L_2 \) is a congruence-preserving extension of \( K_2 \)?

The problem seems to be hard even for finite—rather than congruence-finite—lattices.

For the background of this field, the reader should consult the book [1].

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2. The lattices \( S_p \) and \( T_p \)

Let \( P \) be a finite set with two binary relations, \( \varrho_1 \) and \( \varrho_2 \). For each \( p \in P \), let there be

- a bounded lattice \( S_p \) with a separator \( v_1^p \), that is, with \( 0_p < v_1^p < 1_p \), where \( 0_p \) and \( 1_p \) are the respective zero and unit of \( S_p \);
- a bounded lattice \( T_p \), a sublattice of \( S_p \), with \( |T_p| > 3 \) and with zero \( 0'_p \) and unit \( 1'_p \), that does not contain \( v_1^p \) and that contains a separator \( v_2^p \), with \( 0'_p < v_2^p < 1'_p \), which is doubly irreducible in \( S_p \).

Later on, in our proof of of Theorem 1, we shall have \( 0'_p = 0_p \) but \( 1'_p \neq 1_p \), automatically guaranteeing that \( v_1^p \notin T_p \).

We set \( S_P = \prod(S_p \mid p \in P) \) and \( T_P = \prod(T_p \mid p \in P) \), a subset of \( S_P \). The elements of \( S_P \) will be bold face lower case letters. An element \( s \in S_P \) is written in the form \((s_p)_{p \in P}\). We write \( s_p \) for \( s_p \).

In the papers [5] and [3], a single family of lattices \( S_p, p \in P \), was considered, with a single relation \( \varrho \) on \( P \). If each \( S_p \) is simple, then the direct product of
the lattices \( S_p \) is an isoform lattice whose congruence lattice is boolean. Now, any finite distributive lattice is a sublattice of a finite boolean lattice. In those papers it was shown that, by judicious choice of the relation \( \varrho \), it is possible to define a lattice suborder of the direct product order, whose congruence lattice is a sublattice of the boolean congruence lattice of the direct product, thus yielding an isoform lattice with the correct congruence lattice. In this paper, we extend this this idea. Eventually, we shall take the \( T_p \) as ideals of the \( S_p \), all the \( S_p \) and \( T_p \) simple, and use the two relations \( \varrho_1 \) and \( \varrho_2 \) to get a lattice suborder of the direct product of the \( S_p \). The lattice suborder is a congruence preserving isoform extension of \( K_1 \), with an ideal, which is the induced lattice suborder of the direct product of the \( T_p \); this lattice suborder is a congruence preserving isoform extension of \( K_2 \). Furthermore, the congruences restrict in the desired manner.

We proceed to define the desired order on the direct product \( S_P \). We define a binary relation \( \leq \) on \( S_P \):

**Definition 3.** Let \( a, b \in S_P \). Then \( a \leq b \) iff \( a_p \leq b_p \), for all \( p \in P \), and the following two conditions hold:

- \((P1)\) If \( p \in P \) and \( a_p = v_i^p = b_p \), then \( a_q = b_q \), for all \( q \in P \) with \( p \varrho q \).
- \((P2)\) If \( p \in P \) and \( a_p = v_i^p = b_p \), then \( a_q = b_q \), for all \( q \in P \) with \( p \varrho q \).

**Lemma 4.** The relation \( \leq \) is an order on \( S_P \).

**Proof.** That \( \leq \) is reflexive is immediate, since \((P1)\) and \((P2)\) hold if \( a = b \).

That \( \leq \) is antisymmetric is also immediate, since it is a subrelation of an anti-symmetric relation.

We now establish transitivity. Let \( a \leq b \leq c \). Then, for each \( p \in P \),

\[
(1) \quad a_p \leq b_p \leq c_p,
\]

and so

\[
a_p \leq c_p.
\]

We need only establish \((P1)\) and \((P2)\) for the pair \( a, c \). Let \( i \in \{1, 2\} \), let \( a_p = v_i^p = c_p \), and let \( p \varrho q \), for some \( q \in P \). Then, by \((1)\), \( a_p = v_i^p = b_p \) and \( b_p = v_i^p = c_p \). Thus, by \((P1)\), for \( a \leq b \), we get \( a_q = b_q \), and, similarly for \( b \leq c \), we get \( b_q = c_q \). Thus \( a_q = c_q \), establishing \((P1)\) for \( a, c \), thereby completing the demonstration that \( a \leq c \). Thus \( \leq \) is also transitive.

Since \( v_i^p \notin T_p \), for all \( p \in P \), the following result is immediate:

**Lemma 5.** Let \( a, b \in T_P \). Then \( a \leq b \) iff \( a_p \leq b_p \), for all \( p \in P \) and condition \((P2)\) holds for \( a, b \).

We now proceed to show that \( S_P \) is a lattice under the above defined order \( \leq \). We work out the details for the join operation and appeal to the principle of duality to get the meet operation.

We define a binary operation \( \lor \) on \( S_P \) and then proceed to show that \( a \lor b \) is the least upper bound of \( \{a, b\} \) under \( \leq \).

For some \( p \in P \), we shall have \( (a \lor b)_p = a_p \lor b_p \). However, because of conditions \((P1)\) and \((P2)\), for certain \( p \in P \), which we call \textit{join-singular} with respect to \( a, b \), we are forced to have \( (a \lor b)_p > a_p \lor b_p \). We shall give an inductive definition of join-singularity in the following definition. For each integer \( n \geq 0 \), we shall define \( 1\text{-join-singular}_n \) and \( 2\text{-join-singular}_n \). We shall say that \( p \) is \( 1\text{-join-singular}_n \), if it is either \( 1\text{-join-singular}_n \) or \( 2\text{-join-singular}_n \). We shall say \( p \) is \( i\text{-join singular} \) if it is
Let $q \in 1$-join-singular or 2-join-singular. We proceed with the definition:

**Definition 6.** Let $a, b \in S_P$, let $p \in P$, and let $i \in \{1, 2\}$.

Then $p$ is $i$-join-singular with respect to $a, b$, if $a_p \lor b_p = v_i^p$ and there is an $q \in P$ with $p \lor_i q$ and with either $a_q \leq b_p$ and $a_q \not\leq b_q$ or with $b_p \leq a_q$ and $b_q \not\leq a_q$.

Let $n > 0$. Then $p$ is $i$-join-singular with respect to $a, b$, if $a_p \lor b_p = v_i^p$ and there is an $q \in P$ with $p \lor_i q$ such that $q$ is join-singular, $(n-1)$ (that is, either 1-join-singular or 2-join-singular) with respect to $a, b$.

$p$ is $i$-join-singular with respect to $a, b$, if it is $i$-join-singular with respect to $a, b$ for some integer $n \geq 0$, and is join-regular with respect to $a, b$, if it is either 1-join-singular or 2-join-singular with respect to $a, b$.

If $p$ is neither 1-join-singular nor 2-join-singular with respect to $a, b$, then $p$ is join-regular with respect to $a, b$.

Note that, since both $v_i^p$ and $v_i^p$ are join-irreducible in $S_P$, if $a_p \lor b_p = v_i^p$, then we have either $a_q \leq b_p$ or $b_q \leq a_p$. Note also that, since $v_i^p \neq v_i^q$, 1-join-singularity and 2-join-singularity are mutually exclusive. Furthermore, if $p$ is $i$-join-singular with respect to $a, b$, then $a_q \lor b_p = v_i^p$.

Again, since $v_i^p \not\in T_p$ and $T_p$ is a sublattice of $S_P$, for each $p \in P$, we have:

**Lemma 7.** Let $a, b \in T_P$ and let $p \in P$. Then, for each integer $n \geq 0$, $p$ is join-singular with respect to $a, b$ iff $p$ is 2-join-singular with respect to $a, b$. In particular, $p$ is join-regular with respect to $a, b$ iff $p$ is 2-join-regular with respect to $a, b$.

We now define the binary operation $\lor$ on $S_P$.

**Definition 8.** Let $a, b \in S_P$. Then $a \lor b \in S_P$ is defined by setting

$$(a \lor b)_p = \begin{cases} 1_p, & \text{if } p \text{ is 1-join-singular with respect to } a, b; \\ v_i^p, & \text{if } p \text{ is 2-join-singular with respect to } a, b; \\ a_p \lor b_p, & \text{if } p \text{ is join-regular with respect to } a, b, \end{cases}$$

for each $p \in P$.

From Definition 6, we see that $a_q \lor b_p \leq (a \lor b)_p$, for all $p \in P$.

By Lemma 7, we have:

**Lemma 9.** Let $a, b \in T_P$. Then, for each $p \in P$,

$$(a \lor b)_p = \begin{cases} 1_p, & \text{if } p \text{ is join-singular with respect to } a, b; \\ a_p \lor b_p, & \text{otherwise}. \end{cases}$$

In particular, $a \lor b \in T_P$.

We now proceed to show that $a \lor b$ is the least upper bound in $(S_P, \leq)$ of $\{a, b\}$.

First, a preliminary result:

**Lemma 10.** Let $a, b \in S_P$, let $i \in \{1, 2\}$, and let $p, q \in P$ with $p \lor_i q$. If $a_p \lor_i b_p = v_i^p$ and $p$ is join-regular with respect to $a, b$, then so is $q$.

**Proof.** If $q$ were join-regular for some integer $n \geq 0$, then $p$ would be $i$-join-singular, $n+1$. \qed
Lemma 11. Let \( a, b \in S_P \). Then \( a \lor b \) is an upper bound of \( \{a, b\} \).

Proof. By symmetry, it suffices to show that \( a \leq a \lor b \).

Clearly,

\[
(a_p, b_p, (a \lor b)_p) \leq (a_p, b_p, (a \lor b)_p),
\]

for all \( p \in P \).

To conclude that \( a \leq a \lor b \), we need only show that conditions (P1) and (P2) hold for the pair \( a, a \lor b \).

Let \( i \in \{1, 2\} \), let \( p \in P \) with

\[
a_p = v_i^p = (a \lor b)_p,
\]

and let \( p \leq q \). Since \( v_i^p \neq 1_p, 1'_p \), it follows, by Definition 8, that \( p \) is join-regular with respect to \( a, b \). Thus,

\[
v_i^p = a_p = (a \lor b)_p = a_p \lor b_p,
\]

that is,

\[
b_p \leq a_p.
\]

Since \( p \) is join-regular and \( a_p \lor b_p = v_i^p \), it follows that \( b_q \leq a_q \) (\( p \) is not \( i \)-join-singular—see Definition 6). Thus

\[
a_q = a_q \lor b_q = (a \lor b)_q,
\]

establishing conditions (P1) and (P2) for the pair \( a, a \lor b \).

Before proving that \( a \lor b \) is the least upper bound of \( \{a, b\} \), we prove a preliminary lemma:

Lemma 12. Let \( a, b, c \in S_P \) with \( a, b \leq c \) and let \( p \in P \). If \( p \) is join-singular with respect to \( a, b \), then \( a_p \lor b_p < c_p \).

Proof. We proceed by induction.

Let \( i \in \{1, 2\} \) and let \( p \) be \( i \)-join-singular with respect to \( a, b \). Without loss of generality, we may assume that

\[
b_p \leq a_p = v_i^p
\]

and that there is a \( q \in P \) with \( p \leq q \) and with

(2) \( b_q \not\leq a_q \).

Now, \( a_p = a_p \lor b_p \leq c_p \). Assume that \( a_p \lor b_p = c_p \). Then we have

\[
a_p = v_i^p = c_p.
\]

By condition (Pi) for \( a \leq c \), we get

\[
a_q = c_q,
\]

contradicting (2), since (2) implies that

\[
a_q < a_q \lor b_q \leq c_q.
\]

This contradiction shows that \( a_p \lor b_p < c_p \) if \( p \) is join-singular.
Now, for the induction step, let $n > 0$ and let $a_q \lor b_q < c_q$, for all $q \in P$ that are join-singular$_{n-1}$ with respect to $a, b$. Let $i \in \{1, 2\}$, let $p$ be $i$-join-singular$_n$ with respect to $a, b$, and let us assume that

$$a_p \lor b_p = c_p.$$  
Now, $a_p \lor b_p = v^p_i$, which is join-irreducible. Thus, we may assume that $b_p \leq a_p$.

We then have

$$a_p = a_p \lor b_p = v^p_i = c_p.$$  
Now, since $p$ is $i$-join-singular$_n$, there is a $q \in P$ with $p \preceq q$, and with $q$ join-singular$_{n-1}$. Then

$$a_q \lor b_q < c_q.$$  
But, by condition (Pi) for $a \leq c$ and by (3),

$$c_q = a_q \leq a_q \lor b_q \leq c_q,$$
contradicting (4).

This contradiction establishes the inductive step, thereby proving the lemma. □

**Lemma 13.** For $a, b \in S_P$, the element $a \lor b$ is the least upper bound of $\{a, b\}$.

**Proof.** By Lemma 11, we need only show that if $c \in S_P$ and $a, b \leq c$, then $a \lor b \leq c$.

We first show that

$$(a \lor b)_p \leq c_p,$$
for all $p \in P$. Since $a_p, b_p \leq c_p$, we conclude that

$$(a \lor b)_p = a_p \lor b_p \leq c_p,$$
for all $p$ join-regular with respect to $a, b$.

If $p$ is join-singular with respect to $a, b$, then, by Lemma 12,

$$(a \lor b)_p \lor b_p < c_p.$$  
If $p$ is 1-join-singular, then $a_p \lor b_p = v^p_1$, whereby (5) implies that $c_p = 1_p = (a \lor b)_p$.

If $p$ is 2-join-singular, then $a_p \lor b_p = v^p_2$, whereby (5) implies that $c_p \geq 1_p = (a \lor b)_p$, since $v^p_2$ is meet-irreducible.

Thus $(a \lor b)_p \leq c_p$, for all $p \in P$.

We now establish conditions (P1) and (P2) for the pair $a \lor b, c$.

Let $i \in \{1, 2\}$, let $p \preceq q$, and let $(a \lor b)_p = v^p_i = c_p$. Since $v^p_i \neq 1_p, 1'_{p'}$, we conclude, by Definition 8, that $p$ is join-regular with respect to $a, b$, and so that

$$a_p \lor b_p = (a \lor b)_p = v^p_i = c_p.$$  
Since $v^p_i$ is join-irreducible, we may assume that

$$a_p = a_p \lor b_p = v^p_i = c_p.$$  
Since then $b_p \leq a_p$ and since $p$ is join-regular, we conclude that

$$b_q \leq a_q$$
and, by Lemma 10, $q$ is join-regular. Thus

$$(a \lor b)_q = a_q \lor b_q = a_q.$$  
Since $a_p = v^p_i = c_p$, we get $a_q = c_q$ by condition (Pi) for $a \leq c$, and so

$$(a \lor b)_q = a_q = c_q,$$
establishing condition \((P_i)\) for \(a \lor b\) and \(c\).

Thus, if \(a, b \leq c\), then \(a \lor b \leq c\), proving that \(a \lor b\) is the least upper bound of \(\{a, b\}\).

By duality, we have the concepts of meet-singularity and the meet operation \(\land\) on \(S_P:\)

**Definition 14.** Let \(a, b \in S_P\), let \(p \in P\), and let \(i \in \{1, 2\}\).

Then \(p\) is \(i\)-meet-singular\(_0\) with respect to \(a, b\), if \(a_p \land b_p = v_i^p\) and there is a \(q \in P\) with \(p \uplus q\) and with either \(a_p \leq b_p\) and \(a_q \not\leq b_q\) or with \(b_p \leq a_p\) and \(b_q \not\leq a_q\).

Let \(n > 0\). Then \(p\) is \(i\)-meet-singular\(_n\) with respect to \(a, b\), if \(a_p \land b_p = v_i^p\) and there is a \(q \in P\) with \(p \uplus q\) such that \(q\) is meet-singular\(_{n-1}\) (that is, either \(1\)-meet-regular or \(2\)-meet-singular\(_{n-1}\)) with respect to \(a, b\).

\(p\) is \(i\)-meet-singular with respect to \(a, b\), if it is \(i\)-meet-singular\(_n\) with respect to \(a, b\) for some integer \(n \geq 0\), and is meet-singular with respect to \(a, b\) if it is either \(1\)-meet-singular or \(2\)-meet-singular with respect to \(a, b\).

If \(p\) is neither \(1\)-meet-singular nor \(2\)-meet-singular with respect to \(a, b\), then \(p\) is meet-regular with respect to \(a, b\).

**Definition 15.** Let \(a, b \in S_P\). Then \(a \land b \in S_P\) is defined by setting

\[
(a \land b)_p = \begin{cases} 
0_p, & \text{if } p \text{ is } 1\text{-meet-singular with respect to } a, b; \\
0'_p, & \text{if } p \text{ is } 2\text{-meet-singular with respect to } a, b; \\
a_p \land b_p, & \text{if } p \text{ is meet-regular with respect to } a, b,
\end{cases}
\]

for each \(p \in P\).

We then have:

**Lemma 16.** The order \(S_P\) is a lattice with join operation \(\lor\) of Definition 8 and meet operation \(\land\) of Definition 15. The subset \(T_P\) is a sublattice of \(S_P\).

**Proof.** That \(S_P\) is a lattice under \(\lor\) and \(\land\) follows from Lemma 13 and its dual. That \(T_P\) is a sublattice follows from Lemma 9 and its dual.\(\square\)

### 3. Some Sublattices and Congruences

Let \(Q \subseteq P\). Then, restricting \(g_1\) and \(g_2\) to \(Q\), we have the lattices \(S_Q = \prod(S_p \mid p \in Q)\) and \(T_Q = \prod(T_p \mid p \in Q)\), as in Section 2. Let \(x \in S_{P-Q} = \prod(S_p \mid p \notin Q)\).

We then have a (set) injection \(\eta_x: S_Q \to S_P\) determined by

\[
(\eta_x a)_p = \begin{cases} 
 a_p, & \text{if } p \in Q; \\
x_p, & \text{if } p \notin Q.
\end{cases}
\]

**Lemma 17.** Let \(x \in S_{P-Q}\) with \(x_p \neq v_i^p, v_2^p\), for all \(p \notin Q\). Let \(a, b \in S_Q\), and let \(p \in P\). Then, for each \(i \in \{1, 2\}\), \(p\) is \(i\)-join-singular with respect to \(\eta_x a, \eta_x b\) iff \(p \in Q\) and \(p\) is \(i\)-join-singular with respect to \(a, b\).

**Proof.** Clearly, if \(p \in Q\) and \(p\) is \(i\)-join-singular with respect to \(a, b\), then \(p\) is \(i\)-join singular with respect to \(\eta_x a, \eta_x b\).

Now, let \(i \in \{1, 2\}\) and let \(p \in P\) be \(i\)-join-singular with respect to \(\eta_x a, \eta_x b\).

Since, for \(p \notin Q\),

\[
(\eta_x a)_p \lor (\eta_x b)_p = x_p \lor x_p = x_p = v_i^p,
\]

...
it follows that \( p \in Q \).

It then suffices to prove the following statement, for each integer \( n \geq 0 \):

\((Sn)\) If \( i \in \{1, 2\} \), if \( p \in Q \), and if \( p \) is \( i \)-join-singular with respect to \( \eta_xa, \eta_xb \), then \( p \) is \( i \)-join-singular with respect to \( a, b \).

We proceed by induction.

We first prove (S0):

Let \( i \in \{1, 2\} \), let \( p \in Q \), and let \( p \) be \( i \)-join-singular with respect to \( \eta_xa, \eta_xb \). Thus, \( (\eta_xa)_p \lor (\eta_xb)_p = v_i^p \), that is, \( a_p \lor b_p = v_i^p \), and, say, \( (\eta_xb)_p \leq (\eta_xa)_p \), that is, \( b_p \leq a_p \), and there is a \( q \in P \) with \( p \lor q \), and with \( (\eta_xb)_q \nleq (\eta_xa)_q \). Then \( q \) must lie in \( Q \), since \( q \notin Q \) implies that \( (\eta_xb)_q = x_q = (\eta_xa)_q \). So \( b_q \nleq a_q \). Thus \( p \) is \( i \)-join-singular with respect to \( a, b \), establishing (S0).

Now, for the inductive step, let \( n > 0 \) and assume \((S(n-1))\). Let \( i \in \{1, 2\} \) and let \( p \in Q \) be \( i \)-join-singular with respect to \( \eta_xa, \eta_xb \). Then, \( a_p \lor b_p = v_i^p \), and there is a \( q \) with \( p \lor q \) such that \( q \) is join-singular with respect to \( \eta_xa, \eta_xb \). Then \( q \in Q \) and, by statement \((S(n-1))\), \( q \) is join-singular with respect to \( a, b \). So \( p \) is \( i \)-join-singular with respect to \( a, b \), thereby establishing the inductive step.

Thus statement \((Sn)\) holds for all \( n \geq 0 \), concluding the proof. \( \square \)

**Corollary.** If \( x \in SP-Q \) and \( x_p \neq v_i^1, v_i^2 \), for all \( p \notin Q \), then \( \eta_x \) is a lattice embedding of \( S_Q \) into \( S_P \). If, furthermore, \( x \in TP-Q \), then \( \eta_x \) embeds \( T_Q \) as a sublattice of \( T_P \).

**Proof.** To show that \( \eta_x \) is a lattice embedding, we need only show that \( \eta_x \) preserves join and meet (since \( \eta_x \) is injective). By duality, it suffices to show that \( \eta_x \) preserves join.

Let \( a, b \in S_Q \), let \( p \in P \), and consider \((\eta_xa \lor \eta_xb)_p \). If \( p \notin Q \), then, by Lemma 17, \( p \) is join-regular with respect to \( \eta_xa, \eta_xb \). So

\[(\eta_xa \lor \eta_xb)_p = (\eta_xa)_p \lor (\eta_xb)_p = x_p = (\eta_x(a \lor b))_p,\]

if \( p \notin Q \).

Now let \( p \in Q \) be join-regular with respect to \( \eta_xa, \eta_xb \). Then, by Lemma 17, \( p \) is join-regular with respect to \( a, b \). Then

\[(\eta_xa \lor \eta_xb)_p = (\eta_xa)_p \lor (\eta_xb)_p = a_p \lor b_p = (a \lor b)_p = (\eta_x(a \lor b))_p,\]

if \( p \in Q \) is join-regular with respect to \( \eta_xa, \eta_xb \).

Next, let \( p \in Q \) be 1-join-singular with respect to \( \eta_xa, \eta_xb \). Then \( p \) is 1-join singular with respect to \( a, b \). Then

\[(\eta_xa \lor \eta_xb)_p = 1_p = (a \lor b)_p = (\eta_x(a \lor b))_p,\]

if \( p \in Q \) is 1-join-singular with respect to \( \eta_xa, \eta_xb \).

Finally, let \( p \in Q \) be 2-join-singular with respect to \( \eta_xa, \eta_xb \). Then \( p \) is 2-join singular with respect to \( a, b \). Then

\[(\eta_xa \lor \eta_xb)_p = 1'_p = (a \lor b)_p = (\eta_x(a \lor b))_p,\]

if \( p \in Q \) is 2-join-singular with respect to \( \eta_xa, \eta_xb \).

Consequently, for all \( p \in P \), \((\eta_xa \lor \eta_xb)_p = (\eta_x(a \lor b))_p \). That is,

\[\eta_xa \lor \eta_xb = \eta_x(a \lor b).\]

So, \( \eta_x \) preserves join and, thus, by duality, is a lattice homomorphism.
The last statement in the corollary follows because $T_Q$ is a sublattice of $S_Q$ and $T_P$ is a sublattice of $S_P$ by Lemma 16.

A subset $Q \subseteq P$ is said to be a $g_1$-down-set, if whenever $p \in Q$ and $q \leq p$, then $q \in Q$. We say that $Q$ is a $\{g_1, g_2\}$-down-set, if it is both a $g_1$-down-set and a $g_2$-down-set. For $Q$ a $\{g_1, g_2\}$-down-set, the analog of Lemma 17 holds for arbitrary $x \in S_{P-Q}$:

**Lemma 18.** Let $Q \subseteq P$ be a $\{g_1, g_2\}$-down-set and let $x \in S_{P-Q}$. Let $a, b \in S_Q$ and let $p \in P$. Then, for each $i \in \{1, 2\}$, $p$ is $i$-join-singular with respect to $\eta_xa$, $\eta_xb$ iff $p \in Q$ and $p$ is $i$-join-singular with respect to $a, b$.

**Proof.** It is immediate that if $p \in Q$ and $p$ is $i$-join-singular with respect to $a, b$, then $p$ is $i$-join-singular with respect to $\eta_xa$, $\eta_xb$.

It then suffices to prove the following statement, for each integer $n \geq 0$:

(Sn) If $i \in \{1, 2\}$ and if $p \in P$ is $i$-join-singular$_n$ with respect to $\eta_xa$, $\eta_xb$, then $p \in Q$ and $p$ is $i$-join-singular$_n$ with respect to $a, b$.

We proceed by induction.

We first prove (S0).

Let $i \in \{1, 2\}$, and let $p$ be $i$-join-singular$_0$ with respect to $\eta_xa$, $\eta_xb$. Thus $(\eta_xa)_p \lor (\eta_xb)_p = v^p_i$ with, say, $(\eta_xb)_p \leq (\eta_xa)_p$, and there is a $q \in P$ with $p \leq q$ and with $(\eta_xb)_q \not\leq (\eta_xa)_q$. Then, as in the proof of Lemma 17, $q$ must lie in $Q$. So, since $Q$ is a $g_1$-down-set, $p \in Q$. Then $a_p \lor b_p = v^p_i$, $b_p \leq a_p$, and $b_q \not\leq a_q$. Thus $p$ is $i$-join-singular$_0$ with respect to $a, b$, establishing (S0).

Now for the inductive step, let $n > 0$ and assume (Sn), i.e., $i \in \{1, 2\}$ and let $p$ be $i$-join-singular$_n$ with respect to $\eta_xa$, $\eta_xb$. Then $(\eta_xa)_p \lor (\eta_xb)_p = v^p_i$ and there is a $q \in P$ with $p \leq q$ such that $q$ is join-singular$_{n-1}$ with $\eta_xa$, $\eta_xb$. Then, by (S(n-1)), $q \in Q$ and $q$ is join-singular$_{n-1}$ with respect to $a, b$. Since $Q$ is a $g_1$-down-set, $p$ is then in $Q$ and so $a_p \lor b_p = v^p_i$, $p \leq q$, and $q$ is join-singular$_{n-1}$ with respect to $a, b$; that is, $p$ is $i$-join-singular$_n$ with respect to $a, b$. So the inductive step is established.

Thus (Sn) holds, for all $n > 0$, concluding the proof.

**Corollary.** If $Q \subseteq P$ is a $\{g_1, g_2\}$-down-set and $x \in S_{P-Q}$, then $\eta_x$ is a lattice embedding of $S_Q$ into $S_P$.

**Proof.** The proof is a verbatim copy of the proof of the Corollary to Lemma 17, with all discussion of $T_Q$ and $T_P$ deleted, and with each reference to Lemma 17 replaced with a reference to Lemma 18.

**Lemma 19.** Let $Q \subseteq P$ be a $g_2$-down-set and let $x \in S_{P-Q}$. Let $a, b \in T_Q$ and let $p \in P$. Then $p$ is join-singular with respect to $\eta_xa$, $\eta_xb$ iff $p \in Q$ and $p$ is join-singular with respect to $a, b$.

**Proof.** By Lemma 7, join-singularity with respect to elements $a, b$ with $a_p, b_p \in T_p$, for all $p \in P$, is 2-join-singularity. Our lemma is then almost a consequence of Lemma 18. The only difference is that in Lemma 18 $Q$ is a $\{g_1, g_2\}$-down-set, while here $Q$ is only a $g_2$-down-set. However, because here join-singularity is 2-join-singularity, the relation $g_1$ never appears in the characterization of join-singularity. Thus, here, it suffices that $Q$ be only a $g_2$-down-set.

Then, exactly as for the corollary to Lemma 18, we have the corollary:
Corollary. If \( Q \subseteq P \) is a \( \varrho_2 \)-down-set and \( x \in T_{P - Q} \), then \( \eta_x \) is a lattice embedding of \( T_Q \) into \( T_P \).

We now look at some congruences.

A subset \( Q \subseteq P \) is said to be a \( \varrho_i \)-up-set (for \( i \in \{1, 2\} \)), if \( p \in Q \) and \( \varrho_i q \) imply that \( q \in Q \). We say that \( Q \) is a \( \{\varrho_1, \varrho_2\} \)-up-set if \( Q \) is both a \( \varrho_1 \)-up-set and a \( \varrho_2 \)-up-set.

For each \( Q \subseteq P \), we have the projection \( \pi_Q : S_P \to S_Q \) and its restriction \( \pi_Q' : T_P \to T_Q \) given by \( (\pi_Q a)_p = a_p \) and \( (\pi_Q a)_p = a_p \), for \( p \in Q \).

Then we have the following duals, in a category-theoretic sense, of Lemmas 18 and 19.

Lemma 20. Let \( Q \subseteq P \) be a \( \{\varrho_1, \varrho_2\} \)-up-set and let \( a, b \in S_P \). Then, for each \( p \in Q \) and each \( i \in \{1, 2\} \), \( p \) is \( i \)-join-singular with respect to \( a, b \) iff \( p \) is \( i \)-join-singular with respect to \( \pi_Q a, \pi_Q b \).

Proof. If \( p \in Q \) is \( i \)-join-singular with respect to \( \pi_Q a, \pi_Q b \), then it follows immediately that \( p \) is \( i \)-join-singular with respect to \( a, b \).

It then suffices to prove the following statement, for each integer \( n \geq 0 \):

\( (S_n) \) If \( i \in \{1, 2\} \), \( p \in Q \), and \( p \) is \( i \)-join-singular with respect to \( a, b \), then \( p \) is \( i \)-join-singular with respect to \( \pi_Q a, \pi_Q b \).

We proceed by induction.

We first prove \((S0)\). Let \( i \in \{1, 2\} \), let \( p \in Q \), and let \( p \) be \( i \)-join-singular with respect to \( a, b \). Thus, \( a_p \lor b_p = v_i^p \), that is, \( (\pi_Q a)_p \lor (\pi_Q b)_p = v_i^p \), and, say, \( b_p \leq a_p \), that is, \( (\pi_Q a)_p \leq (\pi_Q b)_p \), and there is a \( q \in P \) with \( \varrho_i q \), and with \( b_q \not\leq a_q \). Then \( q \in Q \), since \( Q \) is a \( \varrho_i \)-up-set, and so \( (\pi_Q b)_q \not\leq (\pi_Q a)_q \). Thus \( p \) is \( i \)-join-singular0 with respect to \( \pi_Q a, \pi_Q b \), establishing \((S0)\).

Now, for the inductive step, let \( n > 0 \) and assume \((S(n - 1))\). Let \( i \in \{1, 2\} \) and let \( p \in Q \) be \( i \)-join-singular with respect to \( a, b \). Then, \( (\pi_Q a)_p \lor (\pi_Q b)_p = a_p \lor b_p = v_i^p \), and there is a \( q \in P \) with \( \varrho_i q \) such that \( q \) is \( \varrho_i \)-up-set, and, by statement \((S(n - 1))\), \( q \) is \( i \)-join-singular \( n - 1 \) with respect to \( a, b \). Then \( q \in Q \), since \( Q \) is a \( \varrho_i \)-up-set, and, by statement \((S(n - 1))\), \( q \) is \( i \)-join-singular \( n - 1 \) with respect to \( \pi_Q a, \pi_Q b \). So \( p \) is then \( i \)-join-singular \( n \) with respect to \( \pi_Q a, \pi_Q b \), thereby establishing the inductive step.

Thus \((Sn)\) holds for all \( n \geq 0 \), concluding the proof.

Corollary. Let \( Q \subseteq P \) be a \( \{\varrho_1, \varrho_2\} \)-up-set. Then \( \pi_Q : S_P \to S_Q \) is a lattice surjection.

Proof. By duality, it suffices to show that \( \pi_Q \) preserves join.

Let \( a, b \in S_P \), let \( p \in Q \), and consider \( (\pi_Q a \lor \pi_Q b)_p \).

If \( p \) is \( \varrho_i \)-regular with respect to \( a, b \), then, by Lemma 20, \( p \) is \( \varrho_i \)-regular with respect to \( \pi_Q a, \pi_Q b \). Then

\[ (\pi_Q a \lor \pi_Q b)_p = (\pi_Q a)_p \lor (\pi_Q b)_p = a_p \lor b_p = (a \lor b)_p = (\pi_Q (a \lor b))_p, \]

if \( p \in Q \) is \( \varrho_i \)-regular with respect to \( a, b \).

Next, let \( p \in Q \) be 1-join-singular with respect to \( a, b \). Then \( p \) is 1-join-singular with respect to \( \pi_Q a, \pi_Q b \). So

\[ (\pi_Q a \lor \pi_Q b)_p = 1_p = (a \lor b)_p = (\pi_Q (a \lor b))_p, \]

if \( p \in Q \) is 1-join-singular with respect to \( a, b \).
Finally, let \( p \in Q \) be 2-join-singular with respect to \( a, b \). Then \( p \) is 2-join-singular with respect to \( \pi_Q a, \pi_Q b \). So

\[
(p_Q a \lor p_Q b)_p = 1_p = (a \lor b)_p = (p_Q (a \lor b))_p,
\]

if \( p \in Q \) is 2-join-singular with respect to \( a, b \).

Consequently, for all \( p \in Q \), \( (p_Q a \lor p_Q b)_p = (p_Q (a \lor b))_p \). That is,

\[
\pi_Q a \lor \pi_Q b = \pi_Q (a \lor b).
\]

So, \( \pi_Q \) preserves join and, thus, by duality, is a lattice homomorphism. \( \square \)

**Lemma 21.** Let \( Q \subseteq P \) be a \( g_2 \)-up-set and let \( a, b \in T_P \). Then, for each \( p \in Q \), \( p \) is join-singular with respect to \( a, b \) iff \( p \) is join-singular with respect to \( \pi'_Q a, \pi'_Q b \).

**Proof.** By Lemma 7, join-singularity here is 2-join-singularity.

If \( p \in Q \) is join-singular with respect to \( \pi'_Q a, \pi'_Q b \), then it follows immediately that \( p \) is join-singular with respect to \( a, b \).

It then suffices to prove the following statement, for each integer \( n \geq 0 \):

\((S_0)\) If \( p \in Q \), and if \( p \) is join-singular, with respect to \( a, b \in T_P \), then \( p \) is join-singular, with respect to \( \pi'_Q a, \pi'_Q b \).

We proceed by induction.

We first prove \((S_0)\).

Let \( p \in Q \), and let \( p \) be join-singular with respect to \( a, b \). Thus, \( a_p \lor b_p = v_2^p \), that is, \((\pi_Q a)_p \lor (\pi_Q b)_p = v_2^p \), and, say, \( b_p \leq a_p \), that is, \((\pi_Q b)_p \leq (\pi_Q a)_p \), and there is a \( q \in P \) with \( p g_2 q \), and with \( b_q \not\leq a_q \). Then \( q \in Q \), since \( Q \) is a \( g_2 \)-up-set, and so \((\pi'_Q b)_q \not\leq (\pi'_Q a)_q \). Thus \( p \) is join-singular with respect to \( \pi'_Q a, \pi'_Q b \), establishing \((S_0)\).

Now, for the inductive step, let \( n > 0 \) and assume that \((S(n-1))\). Let \( p \in Q \) be join-singular, with respect to \( a, b \). Then, \((\pi'_Q a)_p \lor (\pi'_Q b)_p = a_p \lor b_p = v_2^p \), and there is a \( q \) with \( p g_2 q \) such that \( q \) is join-singular, with respect to \( a, b \). Then \( q \in Q \), since \( Q \) is a \( g_2 \)-up-set, and, by statement \((S(n-1))\), \( q \) is then join-singular, with respect to \( \pi'_Q a, \pi'_Q b \). So \( p \) is then join-singular, with respect to \( \pi'_Q a, \pi'_Q b \), thereby establishing the inductive step.

Thus \((S_n)\) holds for all \( n \geq 0 \), concluding the proof. \( \square \)

**Corollary.** Let \( Q \subseteq P \) be a \( g_2 \)-up-set. Then \( \pi'_Q : T_P \to T_Q \) is a lattice surjection.

**Proof.** By duality, it suffices to show that \( \pi'_Q \) preserves join.

Let \( a, b \in T_P \) and \( p \in Q \); consider \((\pi_Q a \lor \pi_Q b)_p \).

If \( p \) is join-regular with respect to \( a, b \), then, by Lemma 21, \( p \) is join-regular with respect to \( \pi'_Q a, \pi'_Q b \). Then

\[
(\pi'_Q a \lor \pi'_Q b)_p = (\pi'_Q a)_p \lor (\pi'_Q b)_p = a_p \lor b_p = (a \lor b)_p = (\pi_Q (a \lor b))_p,
\]

if \( p \in Q \) is join-regular with respect to \( a, b \).

If \( p \in Q \) is join-singular with respect to \( a, b \), then \( p \) is join singular with respect to \( \pi'_Q a, \pi'_Q b \). So

\[
(\pi'_Q a \lor \pi'_Q b)_p = 1_p = (a \lor b)_p = (\pi_Q (a \lor b))_p,
\]

if \( p \in Q \) is join-singular with respect to \( a, b \).
Consequently, for all \( p \in Q \), \((\pi'_Q a \lor \pi'_Q b)_p = (\pi'_Q (a \lor b))_p \). That is,
\[
\pi'_Q a \lor \pi'_Q b = \pi'_Q (a \lor b).
\]
So, \( \pi'_Q \) preserves join and, thus, by duality, is a lattice homomorphism. \( \square \)

For each \( \{q_1, q_2\}\)-down-set \( Q \subseteq P \), the set \( P - Q \) is a \( \{q_1, q_2\}\)-up-set, and so we have the congruence \( \Theta' = \ker \pi_{P-Q} \) on \( S_P \). That is, \( a \equiv b \ (\Theta'Q) \iff a_p = b_p \), for all \( p \notin Q \).

Similarly, for each \( q_2\)-down-set \( Q \subseteq P \), we have the congruence \( \Theta'_Q = \ker \pi_{P-Q} \) on \( T_P \), whereby \( a \equiv b \ (\Theta'_Q) \iff a_p = b_p \), for all \( p \notin Q \).

**Lemma 22.** For each \( \{q_1, q_2\}\)-down-set \( Q \subseteq P \), the congruence \( \Theta'_Q \) is an isoform congruence on \( S_P \).

**Proof.** The congruence classes of \( \Theta'_Q \) are precisely the subsets \( \eta_x(S_Q) \), where \( x \) ranges over \( S_{P-Q} \), which, by the Corollary to Lemma 18, are lattices isomorphic to the lattice \( S_Q \). \( \square \)

Similarly, by the Corollary to Lemma 19, we have:

**Lemma 23.** For each \( q_2\)-down-set \( Q \subseteq P \), the congruence \( \Theta'_Q \) is an isoform congruence on \( T_P \).

We now show that if all the lattices \( S_p \) are simple, then the congruences of \( S_P \) are precisely the congruences \( \Theta_Q \), as \( Q \) ranges over the \( \{q_1, q_2\}\)-down-sets of \( P \). Similarly, if all the lattices \( T_p \) are simple, then the congruences of \( T_P \) are precisely the congruences \( \Theta'_Q \) as \( Q \) ranges over the \( q_2\)-down-sets of \( P \).

We denote by \( \mathbf{0} \) the element \( (0_p)_p \in P \) of \( S_P \), and by \( \mathbf{0}' \) the element \( (0'_p)_p \) of \( T_P \). The elements \( \mathbf{0} \) and \( \mathbf{0}' \) are the zeros of the respective lattices \( S_P \) and \( T_P \). For each \( Q \subseteq P \), we define \( u^Q \in S_P \) by setting
\[
(u^Q)_p = \begin{cases} 1_p, & \text{if } p \in Q; \\ 0_p, & \text{if } p \notin Q. \end{cases}
\]
Note that \( u^\mathbf{0} = \mathbf{0} \). If \( Q \) is the singleton \( \{q\} \), then we write \( u^q \) for \( u^{\{q\}} \). Similarly, \( u'^Q \in T_P \) is defined by setting
\[
(u'^Q)_p = \begin{cases} 1'_p, & \text{if } p \in Q; \\ 0'_p, & \text{if } p \notin Q. \end{cases}
\]
and we write \( u'^q \) for \( u'^{\{q\}} \).

Observe that, just from looking at the order \( \leq \),
\[
(6) \quad u^Q = \bigvee (u^p \mid p \in Q),
\]
and
\[
(7) \quad u'^Q = \bigvee (u'^p \mid p \in Q).
\]

**Lemma 24.** Let \( Q \subseteq P \) be a \( \{q_1, q_2\}\)-down-set, and let \( a \in S_P \). Then, for each \( p \in P \),
\[
(a \lor u'^Q)_p = \begin{cases} 1_p, & \text{if } p \in Q; \\ a_p, & \text{if } p \notin Q. \end{cases}
\]
Lemma 26. Let \( a \lor u^Q \geq u^Q \). So if \( p \in Q \), then \((a \lor u^Q)_p \geq (u^Q)_p = 1_p\), that is, \((a \lor u^Q)_p = 1_p\).

By Definitions 8 and 15, where \( 0 \) here denotes the zero of \( S_{P-Q} \). That is, \((a \lor u^Q)_p = a_p\), if \( p \notin Q \). \( \square \)

Exactly the same way, we have:

Lemma 25. Let \( Q \subseteq P \) be a \( q_2 \)-down-set, and let \( a \in T_P \). Then, for each \( p \in P \),

\[
(a \lor u^Q)_p = \begin{cases} 1_p, & \text{if } p \in Q; \\ a_p, & \text{if } p \notin Q. \end{cases}
\]

We shall also need the following trivial observation:

Lemma 26. Let \( a, b \in S_P \) and let \( p \in P \). If \( a_p \neq b_p \), then \((a \lor b)_p > (a \land b)_p\).

Proof. By Definitions 8 and 15,

\[
(a \lor b)_p \geq a_p \lor a_p > a_p \land b_p \geq (a \land b)_p.
\]

Recall that the \( T_p, p \in P \), were chosen to each contain more than three elements.

Lemma 27. Let \( p \in P \), and let \( S_p \) be a simple lattice. Let \( \Theta \) be a congruence on \( S_P \) such that there are \( a, b \in S_P \) with \( a_p \neq b_p \) and with \( a \equiv b \) (\( \Theta \)). Then \( 0 \equiv u^p (\Theta) \).

Proof. We note that, by the Corollary to Lemma 17, \( \eta_0(S_p) \) (where by slight abuse of notation, \( 0 \) here is the zero of \( S_{P-(p)} \), is a sublattice of \( S_P \) with zero \( 0 \) and unit \( u^p \); hence it is isomorphic to \( S_p \), and so it is simple.

By Lemma 26, we may take \( b < a \). There are two cases to consider: the case where \( a_p \neq v^p_1, v^p_2 \) and the case where \( a_p = v^p_i \), for some \( i \in \{1, 2\} \).

Case 1: \( a_p \neq v^p_1, v^p_2 \). Now, \( p \) is meet-regular with respect to \( a, u^p \) since \( a_p \land (u^p)_p = a_p \neq v^p_1, v^p_2 \). So,

\[
(b \land u^p)_p \leq b_p < a_p = (a \land u^p)_p.
\]

Thus \( b \land u^p \) and \( a \land u^p \) are distinct, they are congruent modulo \( \Theta \), and they lie in the simple sublattice \( \eta_0(S_p) \) with zero \( 0 \) and unit \( u^p \). Therefore,

\[
0 \equiv u^p (\Theta).
\]

Case 2: \( a_p = v^p_i \), for some \( i \in \{1, 2\} \). Then, since \( b_p < a_p \), certainly

\[
b_p \leq 0_p.
\]

Since \( v^p_2 \) is doubly irreducible in the simple lattice \( S_p \) and \( |T_p| > 3 \), there is a \( w \in T_p \) distinct from \( 0'_p, 1'_p \), and \( v^p_2 \). Then \( b_p \lor w = w \) and \( a_p \lor w \geq 1'_p \). Set

\[
\mathbf{w}_q = \begin{cases} w, & \text{if } q = p; \\ 0_q, & \text{otherwise}. \end{cases}
\]

Then, \( b \lor w \leq a \lor w \), \( b \lor w \equiv a \lor w (\Theta) \), and

\[
(b \lor w)_p = w_p = w < 1'_p \leq (a \lor w)_p.
\]

Then, \((a \lor w)_p \neq v^p_i \) for \( i \in \{1, 2\} \). So, replacing \( a \) by \( a \lor w \) and \( b \) by \( b \lor w \), we are back in Case 1, concluding the proof. \( \square \)
Let $\Theta$ be a congruence on $T_P$ such that there are $a, b \in T_P$ with $a_p \neq b_p$ and with $a \equiv b$ ($\Theta$). Then $0' \equiv u^{p}_{(\Theta)}$.

For each $p \in P$, we define $v^{p} \in S_P$ by setting

$$
(v^{p})_q = \begin{cases} v^p_q, & \text{if } q = p; \\ 0_q, & \text{otherwise,}
\end{cases}
$$

and we define $v^{p} \in T_P$ by setting

$$
(v^{p})_q = \begin{cases} v^{p}_q, & \text{if } q = p; \\ 0'_q, & \text{otherwise.}
\end{cases}
$$

**Lemma 29.** Let $p, q \in P$. If $q \not\equiv p$, then $v^q \vee u^p = u^{(p,q)}$.

**Proof.** If $p = q$, then, since $v^p \leq u^p$, we get

$$
v^q \vee u^p = v^p \vee u^p = u^p = u^{(p,q)}.
$$

So we may now assume that $p \neq q$. For each $p' \in P$,

$$
(v^q)_{p'} \vee (u^p)_{p'} = \begin{cases} v^q_{p'}, & \text{if } p' = q; \\ 1_p, & \text{if } p' = p; \\ 0'_{p'}, & \text{otherwise.}
\end{cases}
$$

Thus, $(v^q \vee u^p)_{p'} = (u^{(p,q)})_{p'}$ if $p' \neq q$. Now, $(v^q)_q \vee (u^p)_q = v^q_1$, $q \not\equiv p$, $(u^p)_q \leq (v^q)_q$, and $(u^p)_q \not\leq (v^q)_q$, that is, $q$ is 1-join-singular with respect to $v^q, u^p$. So $(v^q \vee u^p)_q = 1^q = (u^{(p,q)})_q$ also. Thus, $v^q \vee u^p = u^{(p,q)}$. \hfill \square

Similarly, we have:

**Lemma 30.** Let $p, q \in P$. If $q \not\equiv p$, then $v^q \vee u^p = u^{(p,q)}$.

**Lemma 31.** Let $S_P$ be a simple lattice, for each $p \in P$. Then the congruences of $S_P$ are precisely the $\Theta_Q$, as $Q$ ranges over the $\{g_1, g_2\}$-down-sets in $P$.

**Proof.** If $Q$ is a $\{g_1, g_2\}$-down-set, then $\Theta_Q$ is a congruence.

Now, let $\Theta$ be a congruence on $S_P$. Set

$$Q(\Theta) = \{ p \in P \mid a \equiv b \quad (\Theta) \quad \text{and} \quad a_p \neq b_p, \quad \text{for some} \quad a, b \in S_P \}$$

We first show that $Q(\Theta)$ is a $\{g_1, g_2\}$-down-set.

Let $p \in Q(\Theta)$ and let $q \not\equiv p$. By Lemma 27, $0 \equiv u^p \quad (\Theta)$ and so, by Lemma 29, $v^q \equiv u^{(p,q)} \quad (\Theta)$. Since $(v^q)_q = v^p_1 \neq 1_q = (u^{(p,q)})_q$, we conclude that $q \in Q(\Theta)$.

Similarly, let $q \not\equiv p$. By Lemma 28, $0' \equiv u^p \quad (\Theta)$ and so, by Lemma 30, $v^q \equiv u^{(p,q)} \quad (\Theta)$. Since $(v^q)_q = v^p_1 \neq 1'_q = (u^{(p,q)})_q$, we conclude that $q \in Q(\Theta)$.

Thus, $Q(\Theta)$ is a $\{g_1, g_2\}$-down-set, and so we have the congruence $\Theta_{Q(\Theta)}$.

We now show that $\Theta = \Theta_{Q(\Theta)}$.

If $a \equiv b$ ($\Theta$), then, by the definition of $Q(\Theta)$, it follows immediately that $a_p = b_p$, for all $p \not\in Q(\Theta)$, that is, that $a \equiv b \quad (\Theta_{Q(\Theta)})$. Thus $\Theta \leq \Theta_{Q(\Theta)}$.

Now, let $a \equiv b \quad (\Theta_{Q(\Theta)})$. So,

$$a_p = b_p \quad \text{if } p \not\in Q(\Theta).$$
Then, by Lemma 24,
\[ a \lor u^{Q_0} = b \lor u^{Q_0}. \]

By Lemma 27, \( 0 \equiv u^p (\Theta) \), for all \( p \in Q(\Theta) \). Then, by (6), \( 0 \equiv u^{Q_0} (\Theta) \) (note that \( Q(\Theta) \) is finite). Thus,
\[ a \equiv a \lor u^{Q_0} (\Theta) \quad \text{and} \quad b \equiv b \lor u^{Q_0} (\Theta). \]

So, \( a \equiv b (\Theta) \), showing that \( \Theta_{Q(\Theta)} \leq \Theta \).

Thus \( \Theta = \Theta_{Q(\Theta)} \) for the \( \{q_1, q_2\}\)-down-set \( Q(\Theta) \), concluding the proof. \( \square \)

Similarly, we have:

**Lemma 32.** Let \( T_p \) be a simple lattice, for each \( p \in P \). Then the congruences of \( T_p \) are precisely the \( \Theta_Q \) as \( Q \) ranges over the \( \varrho_2\)-down-sets in \( P \).

4. The main technical result

Now let \( P_1 \) and \( P_2 \) be (disjoint) finite sets with relations \( \sigma_1 \) and \( \sigma_2 \), respectively. We also assume that there is a mapping \( \psi: P_2 \to P_1 \) that preserves the relations, that is, that, given \( p, q \in P_2 \) with \( p \sigma_2 q \), then \( (\psi p) \sigma_1 (\psi q) \). We set \( P = P_1 \cup P_2 \), and we define relations \( \varrho_1 \) and \( \varrho_2 \) on \( P \) by setting
\[
\varrho_2 = \sigma_2
\]
and
\[
\varrho_1 = \sigma_1 \cup \sigma_2 \cup \{ (p, \psi(p)), (\psi(p), p) \mid p \in P_2 \}.
\]

Note that \( \varrho_2 \subseteq \varrho_1 \), so the concept of a \( \{\varrho_1, \varrho_2\}\)-down-set is identical to that of a \( \varrho_1\)-down-set.

We assume that we have the lattices \( S_p \) with their sublattices \( T_p \), for \( p \in P \), as above. We also assume that all the lattices \( S_p \) and \( T_p \) are simple. We form the lattices \( S_{P_1} = \prod(S_p \mid p \in P_1) \), \( S_P = \prod(S_p \mid p \in P) \), and \( T_{P_2} = \prod(T_p \mid p \in P_2) \). Now, congruence relations on \( S_{P_1} \) and \( T_{P_2} \) are determined by down-sets of \( P_1 \) under the restriction of \( \varrho_1 \) and \( \varrho_2 \) to \( P_1 \) and of down-sets of \( P_2 \) under the restriction of \( \varrho_2 \) to \( P_2 \), respectively. We use the convention that if \( P' \subseteq P \), then a \( \varrho_1\)-downset \( Q \) of \( P' \) is a subset \( Q \subseteq P' \) that is a down-set of \( P' \) under the restriction of \( \varrho_1 \) to \( P' \).

**Lemma 33.**

(a) If \( Q \) is a \( \varrho_1\)-down-set of \( P \), then \( \psi^{-1}(Q \cap P_1) \) is a \( \varrho_2\)-down-set of \( P_2 \).

(b) If \( Q_1 \) is a \( \varrho_1\)-down-set of \( P_1 \), then \( Q_1 \cup \psi^{-1}(Q_1) \) is a \( \varrho_1\)-down-set of \( P \).

(c) If \( Q \) is a \( \varrho_1\)-down-set of \( P \), then \( Q \cap P_2 = \psi^{-1}(Q \cap P_1) \).

**Proof.**

(a) Let \( p, q \in P_2 \) with \( p \in \psi^{-1}(Q \cap P_1) \) and \( q \varrho_2 p \); we must show that \( q \in \psi^{-1}(Q \cap P_1) \). Now, \( q \varrho_2 p \); thus, \( \psi(q) \varrho_1 \psi(p) \), that is, \( \psi(q) \varrho_1 \psi(p) \). As \( Q \) is a \( \varrho_1\)-down-set of \( P \) and \( \psi(p) \in Q \), it follows that \( \psi(q) \in Q \cap P_1 \). So we conclude that \( q \in \psi^{-1}(Q \cap P_1) \).

(b) Let \( q \varrho_1 p \) with \( p \in Q_1 \cup \psi^{-1}(Q_1) \); we must show that \( q \in Q_1 \cup \psi^{-1}(Q_1) \). First, let \( p \in Q_1 \). Then either \( q \in P_1 \), in which case, \( q \in Q_1 \), since \( Q_1 \) is a \( \varrho_1\)-down-set of \( P_1 \). Or \( q \in P_2 \), so that \( q \varrho_1 p \) implies that \( q = \psi(p) \) and so \( q \subseteq \psi^{-1}(Q_1) \). Second, let \( p \in \psi^{-1}(Q_1) \), so that \( \psi(p) \in Q_1 \). If \( q \in P_1 \), then \( q \varrho_1 p \) implies that \( q = \psi(p) \in Q_1 \). If \( q \in P_2 \), then \( q \varrho_2 p \), so that \( \psi(q) \varrho_1 \psi(p) \) and, therefore, \( \psi(q) \in Q_1 \). We conclude that \( q \in \psi^{-1}(Q_1) \).
(c) First, let $p \in Q \cap P_2$. Since $\psi(p) \not\in P_1$, it follows that $\psi(p) \in Q \cap P_1$, and so $p \in \psi^{-1}(Q \cap P_1)$. Second, let $p \in \psi^{-1}(Q \cap P_1)$, so that $\psi(p) \in Q \cap P_1$. Since $p \in P_1$, $\psi(p)$, we have that $p \in Q$ and so $p \in Q \cap P_2$.

We have lattice embeddings

$$\eta_1 : S_{P_1} \to S_P,$$

given by $\eta_1 = \eta_0$, where $\eta_0$ here denotes the zero of $S_{P_2}$, and

$$\eta_2 : T_{P_2} \to S_P,$$

given by $\eta_2 = \eta_0'$, where $\eta_0'$ here denotes the zero of $T_{P_1}$.

**Theorem 34.** Under the conventions of this section, the lattices $S_P$, $S_{P_1}$, and $T_{P_2}$ are all isomorphic lattices. The embedding $\eta_1 : S_{P_1} \to S_P$ is a congruence-preserving extension. For each $g_1$-down-set $Q$ of $P$,

$$\text{res}(\eta_2)\Theta_Q = \Theta_{\psi^{-1}(Q \cap P_1)},$$

*Proof.* The lattices $S_P$, $S_{P_1}$, and $T_{P_2}$ are all isomorphic by Lemmas 22, 23, 31, and 32. Next, we show that the embedding $\eta_1$ is a congruence-preserving extension. If $Q$ is a $g_1$-down-set of $P$, then $Q \cap P_1$ is a $g_1$-down-set of $P_1$, and

$$Q = (Q \cap P_1) \cup \psi^{-1}(Q \cap P_1),$$

since $Q \cap P_2 = \psi^{-1}(Q \cap P_1)$ by Lemma 33. Furthermore, if $Q_1$ is a $g_1$-down-set of $P_1$, then $Q_1 \cup \psi^{-1}(Q_1)$ is a $g_1$-down-set of $P$. Thus the mapping $Q \mapsto Q \cap P_1$ from the $g_1$-down-sets of $P$ to the $g_1$-down-sets of $P_1$ is a bijection. Now, for each $g_1$-down-set $Q$ of $P$, we have $\text{res}(\eta_1)\Theta_Q = \Theta_{Q \cap P_1}$. Thus, by Lemma 31, $\text{res}(\eta_1)$ is a bijection, that is, $\eta_1$ is a congruence-preserving extension, as claimed.

Finally, if $Q$ is a $g_1$-down-set of $P$, then

$$\text{res}(\eta_2)\Theta_Q = \Theta_{Q \cap P_2} = \Theta_{\psi^{-1}(Q \cap P_1)},$$

since $Q \cap P_2 = \psi^{-1}(Q \cap P_1)$ by Lemma 33. 

□

5. THE PROOF OF THEOREM 2

We recall the duality between finite distributive lattices and finite ordered sets. Given a finite distributive lattice $D$, we consider the ordered set $\text{Join} D$ of join irreducible elements of $D$. There is an isomorphism between $D$ and the lattice of $\leq$-down-sets of $\text{Join} D$, whereby $a \in D$ corresponds to $\{ p \in D \mid p \leq a \}$, and, conversely, the $\leq$-down-set $Q$ corresponds to $\bigvee Q \in D$. Furthermore, given a $\{0,1\}$-preserving homomorphism $\varphi : D_1 \to D_2$ of finite distributive lattices, we have the associated isotone map $\psi : \text{Join} D_2 \to \text{Join} D_1$ given by $\psi p = \bigwedge \{ x \in D_1 \mid \varphi x \geq p \}$. Then $\varphi$ corresponds to $\psi^{-1}$ on the $\leq$-down-sets of $\text{Join} D_1$: for each $a \in D_1$, we have $\varphi a = \bigvee \{ p \in \text{Join} D_1 \mid p \leq a \}$.

Now, under the hypotheses of Theorem 2, let $P_1 = \text{Join} D_1$ with relation $\sigma_1$ the order $\leq$ on $P_1$, and let $P_2 = \text{Join} D_2$ with the relation $\sigma_2$ the order $\leq$ on $P_2$. Then, as above, $\varphi : D_1 \to D_2$ determines $\psi : P_2 \to P_1$ satisfying $(\psi p) \; \sigma_1 \; (\psi q)$, for all $p, q \in P_2$ with $p \; \sigma_2 \; q$. As in Section 4, set $P = P_1 \cup P_2$ with the relations $\varphi_2$ and $\varphi_1$ defined on $P$ by (8) and (9) there.

For each $p \in P$, let $S_p$ be a finite simple lattice with separator $v_p^1$ and with a simple ideal $T_p$ with separator $v_p^2$ such that $v_p^1 \notin T_p$. For example, each $T_p$ can be isomorphic to $M_3$ and each $S_p$ can be isomorphic to the lattice depicted in Figure 1.
We then have the lattices $S_{P_1}$, $S_{P}$, and $T_{P_2}$ of Section 4. Then $\eta_2$ embeds $T_{P_2}$ as an ideal in $S_P$. Also, by Theorem 34, $S_{P}$ and $T_{P_2}$ (as well as $S_{P_1}$) are isoform lattices.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{The lattice $S_P$ for Theorem 2.}
\end{figure}

Now, we have an isomorphism $\varepsilon_0: \text{Con} S_{P_1} \rightarrow D_1$ whereby $\varepsilon_0: \Theta_Q \mapsto \bigvee Q$ for $Q$ a $g_1$-down-set of $P_1$, and an isomorphism $\varepsilon_2: \text{Con} T_{P_2} \rightarrow D_2$, whereby $\varepsilon_2: \Theta'_Q \mapsto \bigvee Q$ for $Q$ a $g_2$-down-set of $P_2$.

By Theorem 34, $\text{res}(\eta_1): \text{Con} S_P \rightarrow \text{Con} S_{P_1}$ is an isomorphism. Thus we have an isomorphism $\varepsilon_1 = \varepsilon_0 \circ \text{res}(\eta_1): \text{Con} S_P \rightarrow D_1$.

Furthermore, for each $g_1$-down-set $Q$ of $P$,

$$\text{res}(\eta_2)\Theta_Q = \Theta'_\psi^{-1}(Q \cap P_1).$$

Then

$$\varepsilon_2\text{res}(\eta_2)\Theta_Q = \varepsilon_2\Theta'_\psi^{-1}(Q \cap P_1) = \bigvee \psi^{-1}(Q \cap P_1) = \varphi \bigvee (Q \cap P_1) = \varphi \varepsilon_0 \Theta_Q \cap P_1 = \varphi \varepsilon_0 \text{res}(\eta_1)\Theta_Q = \varphi \varepsilon_1 \Theta_Q.$$

Since any congruence $\Theta$ of $S_P$ is of the form $\Theta_Q$, for some $g_1$-down-set $Q$ of $P$, we thereby have

$$\varepsilon_2\text{res}(\eta_2)\Theta = \varphi \varepsilon_1 \Theta,$$

for all $\Theta \in \text{Con} S_P$. Setting $L_1 = S_P$, $L_2 = T_{P_2}$, and $\eta = \eta_2$, we get Theorem 2.

6. The proof of Theorem 1

For each $i \in \{1, 2\}$, set $P_i = \text{Join}(\text{Con} K_i)$, the ordered set of join-irreducible congruences of $K_i$. Then the $\{0, 1\}$-preserving homomorphism $\varphi: \text{Con} K_1 \rightarrow \text{Con} K_2$ determines, as in Section 5, an isotone map $\psi: P_2 \rightarrow P_1$ with

$$\varphi \Theta = \bigvee \psi^{-1}\{p \in P_1 \mid p \leq \Theta\},$$

for each $\Theta \in \text{Con} K_1$.

For each $p \in P_i$, set

$$(11) \quad \Psi(p) = \bigvee (\Theta \in \text{Con} K_i \mid p \nleq \Theta).$$

Clearly, $p \nleq \Psi(p)$.

Lemma 35. For each $p \in P_i$, the congruence $\Psi(p) \in \text{Con} K_i$ is meet-irreducible.
Proof. Let $\Theta_1, \Theta_2 \in \text{Con} K_i$ with $\Psi(p) < \Theta_1, \Theta_2$. Then there are $p_1, p_2 \in P_i$ with $p_1 \leq \Theta_1, p_2 \leq \Theta_2$, and with $p_1, p_2 \not\in \Psi(p)$. But, then, $p \leq p_1, p_2$, that is, $p \leq p_1 \wedge p_2 \leq \Theta_1 \wedge \Theta_2$. But $p \not\in \Psi(p)$. Thus $\Psi(p) \neq \Theta_1 \wedge \Theta_2$. \hfill $\square$

Lemma 36. For each congruence $\Theta$ of $K_i$,
$$\Theta = \bigwedge (\Psi(p) | \ p \in P_i, p \not\in \Theta).$$

Proof. By (11), $\Theta \leq \Psi(p)$ for all $p \not\in \Theta$, that is,
$$\Theta \leq \bigwedge (\Psi(p) | \ p \in P_i, p \not\in \Theta).$$

Conversely, let $q \in P_i$, and let $q \leq \bigwedge (\Psi(p) | \ p \in P_i, p \not\in \Theta)$, that is, $q \leq \Psi(p)$, for all $p \not\in \Theta$. Thus, for each $p \not\in \Theta$ and all $\Psi \in \text{Con} K_i$ with $p \not\in \Psi$, we have $q \leq \Psi$. Then, taking $\Psi = \Theta$, we get $q \leq \Theta$. Thus,
$$\bigwedge (\Psi(p) | \ p \in P_i, p \not\in \Theta) \leq \Theta$$
as well. \hfill $\square$

Lemmas 35 and 36 are standard results in the duality theory of finite distributive lattices, and the mapping $\Psi$ is the standard isomorphism between the ordered set of join-irreducible elements of a finite distributive lattice and the ordered set of its meet-irreducible elements.

By Lemma 36,

Lemma 37. If $\Theta \in \text{Con} K_i$ and $a, b \in K_i$, then $a \equiv b$ $(\Theta)$ iff $a \equiv b$ $(\Psi(p))$, for all $p \in P_i$ with $p \not\in \Theta$.

By Lemma 35, for each $i \in \{1, 2\}$ and each $p \in P_i$, the congruence $\Psi(p)$ is meet-irreducible and the quotient lattice $K_{ip} = K_i/\Psi(p)$ is thus subdirectly irreducible. It is then easy to embed $K_{ip}$ in a bounded simple lattice $T_p$ with a separator $v^p_2$, as follows. We add a zero $0'_p$ and a unit $1'_p$, and set $a \equiv b$ such that any nontrivial congruence collapses $a$ and $b$. We add a new element $c$ with $0'_p < c < a$, and add two separators $d, v_2^p$ with $0'_p < d, v_2^p < 1'_p$ (see Figure 2.). Then $T_p$ is a simple extension of $K_{ip}$ with a separator $v_2^p$.

We next extend $T_p$ to get $S_p$ by adding a new unit $1_p$ and separators $e, v_1^p$ with $0'_p < e, v_1^p < 1_p$. Then $S_p$ is a simple lattice with $0_p = 0'_p$ and unit $1_p > 1'_p$ (see Figure 3.) that contains $T_p$ as an ideal.

As in Section 5, set $\sigma_1$ on $P_1$ to be the order $\leq$ on $P_1$, and set $\sigma_2$ on $P_2$ to be the order $\leq$ on $P_2$. We set $P = P_1 \cup P_2$ and define the relations $\varrho_2$ and $\varrho_1$ on $P$ by (8) and (9) of Section 4. We then have the lattices $S_{P_1}, S_p, T_{P_2}$, and lattice embeddings $\eta_1: S_{P_1} \to S_P$ and $\eta_2: T_{P_2} \to S_P$, where $\eta_2$ embeds $T_{P_2}$ as an ideal of $S_P$. By Theorem 34, $S_p$ and $T_{P_2}$ (as well as $S_{P_1}$) are isomorphic lattices, $\eta_1$ is a congruence-preserving extension, and

$$\text{res}(\eta_2)\Theta_Q = \Theta^\prime_\psi^{-1}(\Theta \cap P_1),$$

for each $\varrho_1$-down-set $Q$ of $P$.

Now, we have the mappings
$$\text{Diag}_1: K_1 \to S_{P_1},$$
defined by setting $(\text{Diag}_1 a)_p = a/\Psi(p) \in K_{ip} \subseteq S_p$, for each $p \in P_1$, and
$$\text{Diag}_2: K_2 \to T_{P_2},$$
defined by setting $(\text{Diag}_2 a)_p = a/\Psi(p) \in K_{2p} \subseteq T_p$, for each $p \in P_2$.

**Lemma 38.** The mapping $\text{Diag}_1: K_1 \to S_{P_1}$ is a congruence-preserving extension whereby

\[(13) \quad \text{res}(\text{Diag}_1)\Theta_Q = \bigvee Q \in \text{Con} K_1,
\]

for each $\varrho_1$-down-set $Q$ of $P_1 = \text{Join} (\text{Con} K_1)$.

**Proof.** First, $\text{Diag}_1$ is a homomorphism. Since $(\text{Diag}_1 a)_p \in K_{1p}$, for all $p \in P_1$, and $v_1^p, v_2^p \notin K_{1p}$, each $p \in P_1$ is join-regular and meet-regular with respect to $\text{Diag}_1 a$,.
Diag$_1$, b, for any $a, b \in K_1$. Thus, for all $p \in P_1$,

$$(\text{Diag}_1 a \lor \text{Diag}_1 b)_p = (\text{Diag}_1 a)_p \lor (\text{Diag}_1 b)_p = a/\Psi(p) \lor b/\Psi(p) = (a \lor b)/\Psi(p) = (\text{Diag}_1(a \lor b))_p.$$ 

That is, Diag$_1$ preserves $\lor$. Thus, by duality, Diag$_1$ is a homomorphism.

Next, each congruence of $S_{P_1}$ is of the form $\Theta_Q$ for a uniquely determined $\varrho_1$-down-set $Q$ of $P_1$, and, for $a, b \in K_1$,

$$\text{Diag}_1 a \equiv \text{Diag}_1 b(\Theta_Q)$$

iff

$$a \equiv b(\Psi(p)), \text{ for all } p \in P_1 \text{ with } p \notin Q$$

iff

$$a \equiv b(\Psi(p)), \text{ for all } p \in P_1 \text{ with } p \nleq \bigvee Q$$

iff

$$a \equiv b(\bigvee Q)$$

by Lemma 37, thereby establishing (13). Now, res(Diag$_1$) is then surjective since $\Theta = \bigvee(\{p \in P_1 \mid p \leq \Theta\})$, for each $\Theta \in \text{Con} K_1$, and res(Diag$_1$) is injective since there is only one $\varrho_1$-down-set $Q$ of $P_1$ with $\Theta = \bigvee Q$, the set $Q = \{p \in P_1 \mid p \leq \Theta\}$. Thus Diag$_1$ is a congruence-preserving extension, concluding the proof. □

By virtually the same proof, we have:

**Lemma 39.** The mapping Diag$_2$: $K_2 \rightarrow T_{P_2}$ is a congruence-preserving extension whereby

$$\text{res(Diag}_2)\Theta_Q' = \bigvee Q \in \text{Con} K_2,$$

for each $\varrho_2$-down-set $Q$ of $P_2 = \text{Join(Con} K_2)$. 

Now, set

$$L_1 = S_P,$$

and set

$$L_2 = T_{P_2}.$$ 

Then, as noted above, $L_1$ and $L_2$ are isoform lattices.

Define $\varepsilon_1$: $K_1 \rightarrow L_1$, by setting $\varepsilon_1 = \eta_1 \circ \text{Diag}_1$, define $\varepsilon_2$: $K_2 \rightarrow L_2$, by setting $\varepsilon_2 = \text{Diag}_2$, and define $\eta$: $L_2 \rightarrow L_1$, by setting $\eta = \eta_2$. Then $\varepsilon_1$ is a congruence-preserving extension, since $\eta_1$ and Diag$_1$ both are, and $\varepsilon_2 (= \text{Diag}_2)$ is a congruence-preserving extension. Furthermore, $\eta$ embeds $L_2$ as an ideal in $L_1$.

Now, let $\Theta \in \text{Con} L_1$. Then $\Theta = \Theta_Q$, for a uniquely determined $\varrho_1$-down-set $Q$ of $P$. Then

$$\text{res(}\varepsilon_2)\text{res(}\eta)\Theta = \text{res(}\varepsilon_2)\text{res(}\eta_2)\Theta_Q = \text{res(}\varepsilon_2)\Theta_Q' = \text{res}(\Theta_Q'_{\psi^{-1}(Q \cap P_1)})$$

by (12), and

$$\text{res(}\varepsilon_2)\Theta_Q' = \text{res(Diag}_2)\Theta_Q'_{\psi^{-1}(Q \cap P_1)} = \bigvee \psi^{-1}(Q \cap P_1)$$
by (14) of Lemma 39. Now,
\[ \bigvee \psi^{-1}(Q \cap P_1) = \varphi \bigvee (Q \cap P_1) = \varphi \text{res}(\text{Diag}_1) \Theta_{Q \cap P_1} \]
by (13) of Lemma 38. Calculating further,
\[
\text{res}(\text{Diag}_1) \Theta_{Q \cap P_1} \\
= \text{res} (\text{Diag}_1) \text{res}(\eta_1) \Theta_Q \\
= \text{res}(\eta_1 \circ \text{Diag}_1) \Theta_Q \\
= \text{res}(\varepsilon_1) \Theta_Q \\
= \text{res}(\varepsilon_1) \Theta.
\]
Thus,
\[ \text{res}(\varepsilon_2) \text{res}(\eta) \Theta = \varphi \text{res}(\varepsilon_1) \Theta, \]
for all \( \Theta \in \text{Con} L_1 \), concluding the proof of Theorem 1.

References