Which freely generated lattices contain $F(3)$?

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Abstract. In 1979, I. Rival and R. Wille characterized which orders freely generate a lattice that contains $F(3)$, the free lattice on three generators. Their characterization is that the order contains $1+1+1$, $2+3$, or $1+5$ (where $n$ is the $n$-element chain). We give a new proof of their result. In fact, we generalize their result to $m$-lattices (where joins and meets of nonempty sets of cardinality less than $m$ are allowed).

In our proof, we apply a new result; namely, that our lattice-attachment construction preserves breadth when the skeleton is a chain.

1. Introduction

The letter $m$ will always denote an infinite regular cardinal. By an $m$-set, we mean a nonempty set of cardinality less than $m$. An $m$-lattice is a lattice in which joins and meets of $m$-sets can be formed. For an order $P$, let $F_m(P)$ denote the free $m$-lattice on $P$ or the $m$-lattice freely generated by $P$. (We omit $m$ when it is $\aleph_0$.)

Recall that the free $m$-lattice on $n$ generators is the free $m$-lattice on the $n$-element antichain.

An order is linearly decomposable if it is the linear sum of two suborders. Any order is the linear sum of its linearly indecomposable components. Henceforth, we write “indecomposable” instead of “linearly indecomposable.”

Let $n$ denote the $n$-element chain. We call an order slender, if it does not contain $1+1+1$, $2+3$, or $1+5$ as a suborder. The slender order $H$ of six elements is shown in Figure 1 in two ways, as an order and as a “labelled” order. The lattice $F(H)$ was described by I. Rival and R. Wille [7]. The present authors described $F_m(H)$ for arbitrary $m$ in [5].

![Figure 1. The order $H$ with and without labels](image)

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We define the slender order $U$ to consist of two disjoint chains $A$ and $B$, each isomorphic to the integers, both labelled with the integers, with extra comparabilities as follows: if $a \in B$ and $b \in B$ have labels $m$ and $n$, respectively, then $a < b$ iff $m + 2 \leq n$ and $a > b$ iff $m \geq n + 2$. (Compare this with the ordering and the labels in Figure 1.) Clearly, $U$ contains infinitely many copies of $H$.

**Main Theorem.** The following five conditions are equivalent for an order $P$.

(i) $P$ is slender.

(ii) $F_m(Q)$ is isomorphic to an $m$-sublattice of $F_m(U)$ for each indecomposable component $Q$ of $P$.

(iii) $F_m(Q)$ is isomorphic to an $m$-sublattice of $F_m(H)$ for each indecomposable component $Q$ of $P$.

(iv) $F_m(P)$ has breadth at most three.

(v) $F_m(P)$ does not contain $F(3)$ as a sublattice.

Without condition (ii), I. Rival and R. Wille [7] proved the above result for finite $P$ and $m = \aleph_0$. In their condition corresponding to (iii), “indecomposable component” is not needed (because of the finiteness of $P$). Since $F(3)$ is a sublattice of $F_m(3)$, condition (v) of the theorem implies that $F_m(P)$ does not contain $F_m(3)$ as an $m$-sublattice. For $m = \aleph_0$, observe that the equivalence of conditions (i) and (v) for arbitrary orders follows from the finite case.

After the basic results of Section 2, we prove the implication (i)⇒(ii) in Sections 3 and 4. Before discussing this implication, we indicate the remainder of the proof. In Section 5, we prove that $F_m(U)$ is an $m$-sublattice of $F_m(H)$, which proves (ii)⇒(iii). In Section 6, we use attachments to show that $F_m(H)$ has breadth three which, together with Lemma 2.4, proves (iii)⇒(iv). The implication (iv)⇒(v) is true because $F(3)$ has infinite breadth. The implication (v)⇒(i) is due to H. L. Rolf [8]; he gave three independent elements of $2 + 3$ and $1 + 5$.

We introduced the concept of attaching orders in [6]. If $P$ and $Q$ are two orders that agree on their intersection, we describe $R = P[Q]$, the result of attaching $Q$ to $P$. As a set, $R$ is $P \cup Q$. The ordering $\leq_R$ is the least ordering containing $\leq_P$ and $\leq_Q$. We call the order $P \cap Q$ the skeleton. See Figure 2 for a small example.

In general, many orders, called attachments, can be attached simultaneously to the same skeleton. For elements $a$ and $b$ in different attachments, $a < b$ holds in the attachment construction iff $a \leq x$ and $y \leq b$ in the two attachments and

![Figure 2. An attachment example](image-url)
$x \leq y$ holds in the skeleton. The construction contains all the attachments and the skeleton as suborders.

In Theorem 6.5, we show that our lattice-attachment construction preserves breadth when the skeleton is a chain. We apply Theorem 6.5 to show that $F_m(H)$ has breadth three.

The lattice $P\{Q\}$ in Figure 2 can be obtained by attaching the lattices $P$ and $Q$ to the 4-element boolean lattice. Observe that breadth is not preserved in Figure 2.

Let $U^*$ be the disjoint union of two chains $A^*$ and $B^*$, each labelled by half-integers. If $a \in A^*$ and $b \in B^*$ have the half-integer labels $m$ and $n$, respectively, then we define $a < b$ iff $\lceil m \rceil + 2 \leq \lfloor n \rfloor$ and we define $a > b$ iff $\lfloor m \rfloor \geq \lceil n \rceil + 2$. Equivalently, $U^*$ is formed by attaching $A^*$ and $B^*$ to $U$. Clearly, $U$ is a suborder of $U^*$.

We call a suborder $P$ of $U^*$ special, if whenever the non-integer label $n$ occurs in $P$, then none of the labels $n - 1$, $n - \frac{1}{2}$, $n + \frac{1}{2}$, $n + 1$ occur on the same side of $P$. Figure 3 shows such an example. Clearly, any special suborder of $U^*$ is slender.

![Figure 3. A slender order](image-url)

![Figure 4. The orders $R$ and $S$](image-url)
By a pivot of an order, we mean the 1 of a $1 + 4$ suborder. Certainly, any pivot of a suborder of $U^*$ must have a non-integer label. Observe that the element with label 3.5 in Figure 3 is not a pivot. Moreover, this example is not a suborder of $U$.

Although the order $S$ in Figure 4 is slender, it is not a special suborder of $U^*$. In Theorem 3.6, we prove that certain indecomposable slender orders that do not contain $S$ are special suborders of $U^*$. In Section 4, we apply this result to prove the implication (i)$\Rightarrow$(ii).

2. Indecomposable orders and free $m$-lattices

While it is folklore that every order is the linear sum of its linearly indecomposable suborders, we start this section with a proof that lends itself well to the specialization we state in Lemma 2.2.

**Lemma 2.1.** For an order $P$, let $\Theta$ be the equivalence relation on the set $P$ that is generated by the incomparable pairs of $P$. Let $C = P/\Theta$ be the corresponding partition. The set $C$ is a chain whose strict ordering is defined by: $X < Y$ iff $X \neq Y$ and $x < y$, for some $x \in X$ and $y \in Y$. As a suborder of $P$, each element of $C$ is indecomposable. Hence, $P$ is the linear sum of the indecomposable orders in $C$ over the chain $C$.

**Proof.** Assume that $x \Theta y$ and $x < u < y$. There is a finite sequence $x = t_0, t_1, t_2, \ldots, t_n = y$ such that $t_{i-1} \parallel t_i$ for $1 \leq i \leq n$. Suppose that $u$ is comparable with every $t_i$. Let $k$ be the largest value such that $t_k \leq u$. Clearly, $k < n$. From $u < t_{k+1}$, we conclude that $t_k < t_{k+1}$, a contradiction. Therefore, $u \parallel t_i$, for some $i$, so every set in $C$ is convex (as a suborder of $P$).

Let $X$ and $Y$ be distinct elements of $C$. Certainly, $x$ and $y$ are comparable for any $x \in X$ and $y \in Y$. We write $x \varrho Y$ when $x < y$, for some $x \in X$ and $y \in Y$. For $x \in X$ and $y \in Y$, suppose that $x < y$ but $y < u$, for some $u \in X$. Then $x \Theta y$ by convexity, a contradiction. By symmetry, $\varrho$ is an antisymmetric binary relation on $C$. Consequently, $\varrho$ is a strict linear order relation.

Suppose that some $X \in C$ is linearly decomposable. Since $X$ is convex, we can divide $X$ in two parts to give an equivalence relation $\Phi$ properly contained in $\Theta$ such that $P/\Phi$ is a chain. The relation $\Phi$ would then contain all the incomparable pairs of $P$, a contradiction. Therefore, each set in $C$ is indecomposable. \qed

We now specialize Lemma 2.1 to orders of width two.

**Lemma 2.2.** If $P$ is an indecomposable order of width two, then $P$ is the disjoint union of two chains $C$ and $D$. Whenever $x < y$ in $C$, there are elements $c_0, c_1, \ldots, c_n$ of $C$ and $d_1, d_2, \ldots, d_n$ of $D$ such that

$$x = c_0 < c_1 < \cdots < c_n = y$$

and, for $1 \leq i \leq n$, $c_{i-1} \parallel d_i$ and $c_i \parallel d_i$. In fact, one can also assume that

$$d_1 < d_2 < \cdots < d_n.$$
Proof. The first sentence follows by Dilworth’s Theorem [2]. By Lemma 2.1, there are \( x = x_0, x_1, \ldots, x_m = y \) and \( z_1, \ldots, z_m \) with
\[
x_0 \parallel z_1 \parallel x_1 \parallel z_1 \parallel \cdots \parallel z_m \parallel x_m.
\]
Clearly, each \( x_i \) is in \( C \) and each \( z_i \) is in \( D \). We induct on \( m \). Choose the first value of \( k \) such that \( x_k > x \). Clearly, \( k > 0 \) and \( x \parallel z_k \). Let \( c_1 = x_k, d_1 = z_k \) and apply induction to the inequality \( x_k \leq y \).

Observe that the two representing chains of Lemma 2.2 are uniquely determined by \( P \). (Only the argument at the beginning of the proof is needed.) Henceforth, we call each representing chain a “side.” By Lemma 2.2, any interval \([x, y]\) in either side of an indecomposable slender order is finite.

When \( a \) is covered by \( b \) in an order, we write the set \( \{a, b\} \) as \( ab \) and call it a cover or an edge. An edge of either side of indecomposable slender order of width two will be called a “vertical edge.”

P. Crawley and R. A. Dean [1] showed that \( F_m(P) \) is the unique \( m \)-lattice \( L \) that is \( m \)-generated by \( P \) and satisfies the following three conditions:

\( (G_v) \) If \( p \leq \vee Q \) in \( L \) for \( p \in P \) and an \( m \)-subset \( Q \) of \( P \), then \( p \leq q \) for some \( q \in Q \).

\( (G_l) \) If \( \wedge Q \leq p \) in \( L \) for \( p \in P \) and an \( m \)-subset \( Q \) of \( P \), then \( q \leq p \) for some \( q \in Q \).

\( (W_m) \) If \( \wedge X \leq \vee Y \) in \( L \) for \( m \)-subsets \( X \) and \( Y \) of \( L \), then \( x \leq \vee y \) for some \( x \in X \) or \( \wedge x \leq y \) for some \( y \in Y \).

In the above conditions, “\( G \)” stands for “generators” and \( (W_m) \) is the \( m \)-ary extension of the usual Whitman Condition as given, for example, in [3]. A common technique is to construct a “plausible candidate” for \( F_m(P) \) and then show that it satisfies the above conditions. The verification is easiest when we have an \( m \)-sub-lattice of an \( m \)-lattice that satisfies \( (W_m) \).

We say that a set \( S \) generates the downset \( \downarrow S = \{ x \mid x \leq s \text{ for some } s \in S \} \). For an order \( P \), we write \( P^m \) for the extension of \( P \) consisting of all downsets generated by \( m \)-sets, where each \( x \in P \) is identified with \( \downarrow x \) and the order relation is inclusion. By regularity, \( P^m \) is closed under \( m \)-joins. Observe that \( P^m \) is a chain whenever \( P \) is. The dual extension of \( P \) is written as \( P_m \).

**Lemma 2.3.** Let \( C \) be a chain and create \( D \) by attaching \( C^m \) and \( C_m \) to \( C \). We form \( F_m(C) \) from \( D \) by adding the comparability \( a < b \) whenever \( a \parallel b \) in \( D \), \( a \in P^m \) and \( b \in P_m \). Moreover, all the added comparabilities are covers and \( F_m(C) \) is a chain.

*Proof.* The Crawley-Dean conditions are easily verified. \( \square \)

For example, when \( C \) is the chain of rational numbers and \( m \) is uncountable, \( F_m(C) \) is a linear sum over the 2-point compactification of the reals with 3-element chains at rationals and 2-element chains at irrationals.

**Lemma 2.4.** Let \( C \) be a chain. If \( P \) is a linear sum of the orders \( Q_t \), for \( t \in C \), then \( F_m(P) \) is obtained from \( F_m(C) \) by replacing each \( t \in C \) by \( F_m(Q_t) \).

*Proof.* The Crawley-Dean conditions are easily verified. \( \square \)
3. Special suborders

In this section we prove some preliminary results on slender orders.

**Lemma 3.1.** In any $1 + 4$ suborder of a slender order, neither middle element of the 4 is a pivot. In fact, each middle element is incomparable with at most two elements.

**Proof.** Let $\{t\} \cup \{a < b < c < d\}$ be isomorphic to $1 + 4$. By symmetry, it suffices to prove the stronger statement for $b$. Let $r \prec s \prec t \prec u$. To exclude $2 + 3$, $b < u$ and $r < b$. Therefore, $b$ is incomparable with at most $s$ and $t$. □

**Lemma 3.2.** Any indecomposable slender order can be embedded in an unbounded one.

**Proof.** Let $P$ be an indecomposable slender order. Assuming that $P$ is bounded above, let $ab$ and $cd$ be the edges at the top of each side. (If there are no such edges, then $P$ has at most 5 elements. Add at most three elements to $P$.) Clearly, $b \parallel d$. When $ab \parallel cd$, we add a new element at the top of each chain to form, with $\{a, b, c, d\}$, the order $H$. Clearly, the two new vertical edges are incomparable.

If the edges $ab$ and $cd$ are comparable, then we can assume, by symmetry, that $a < d$ and $b \parallel c$. (If $c < b$ also held, $P$ would be linearly decomposable.) Figure 5 shows how to add new elements in two cases.

Case (i): $c$ is a pivot. By Lemma 3.1, $a$ is not a pivot. If $b$ were a pivot, there would be a $2 + 3$ suborder. Thus, adding the two solid elements shown in (i) produces an extension of $P$ that is slender and indecomposable.

Case (ii): $c$ is not a pivot. Thus, adding the solid element shown in (ii) produces an indecomposable slender order.

After the above insertions, there is a $2 + 2$ suborder at the top, a situation that we handled at the beginning. Thus, $P$ can be extended upwards by creating copies of $H$. By this argument and its dual, the result follows. □

Observe that any $2 + 2$ or $1 + 4$ suborder of a slender order is cover-preserving. We call an order *saturated* if it is slender, indecomposable, unbounded in both directions, and each vertical edge is in a $2 + 2$ or $1 + 4$ suborder.
Lemma 3.3. Any indecomposable slender order can be embedded in a saturated order.

Proof. Let $P$ be an indecomposable slender order, which we can assume to be unbounded by Lemma 3.2. Let $ab$ be a vertical edge of $P$ that is not in any $2 + 2$ or $1 + 4$. Since $P$ is indecomposable, there is a single vertex $c$ that is incomparable with $ab$. In Figure 6, the vertex $t$ is inserted on the edge $ab$ to form the order $Q$ which extends $P$. Since $c$ is the only element of $Q$ that is incomparable with $t$, therefore, $Q$ is slender. Since $ac$, we also have that $Q$ is indecomposable. We continue to insert elements in this way until every vertical edge is in some $2 + 2$ or $1 + 4$. Since every vertical edge of $P$ can accept at most two insertions, the process terminates. □

Lemma 3.4. Let $C$ and $D$ be the sides of the saturated order $P$ and let $c_0 < c_1 < c_2$ in $C$ and $d_0 < d_1 < d_2$ in $D$. If $c_0c_1 \parallel d_0d_1$ and neither $c_1$ nor $d_1$ is a pivot, then the six elements form a suborder isomorphic to $H$.

Proof. Observe that $c_0 < d_2$ and $d_0 < c_2$ by exclusion of $2 + 3$. The order $P$ would be linearly decomposable if both $c_1 < d_2$ and $d_1 < c_2$ held. By symmetry, we can assume that $c_2 \parallel d_1$. We are done if $c_1 \parallel d_2$. Therefore, we assume that $c_1 < d_2$. Observe that $d_1$ is the only element of $P$ that is incomparable with $c_1c_2$. Since $d_1$ is not a pivot, $c_1c_2$ is part of a $2 + 2$. Let $uv$ be the edge of $D$ that is incomparable with $c_1c_2$. If $u \geq d_2$, then $c_1 < u$, a contradiction. If $u \leq d_0$, then $u < c_2$, a contradiction. Therefore, $u = d_1$ so that $uv = d_1d_2$. Therefore, $c_1 \parallel d_2$, contradicting our assumption that $c_1 < d_1$. □

Lemma 3.5. If $P$ is a saturated order that does not contain $1 + 4$, then $P \cong U$.

Proof. Let $C$ and $D$ be the sides of $P$. Each edge of $C$ or $D$ is part of a $2 + 2$ suborder. Let $c_0c_1 \parallel d_0d_1$, with $c_0 < c_1$ in $C$ and $d_0 < d_1$ in $D$. We shall inductively define a map $\varepsilon : C \cup D \rightarrow \mathbb{Z}$ that corresponds to an order isomorphism. Recall that each side of $U$ is isomorphic to the integers. We map $C$ to the left side of $U$ and $D$ to the right side. Let $\varepsilon(c_0) = 0$, $\varepsilon(c_1) = 1$, $\varepsilon(d_0) = 0$ and $\varepsilon(d_1) = 1$. So far, we have an order-isomorphism. Let $c_1 < c_2$ in $C$ and $d_1 < d_2$ in $D$. By Lemma 3.4, $c_1c_2 \parallel d_1d_2$. We extend $\varepsilon$ by defining $\varepsilon(c_2) = \varepsilon(d_2) = 2$. Since $c_0 < d_2$ and $d_0 < c_2$ by exclusion of $2 + 3$, we continue to have an isomorphism. Continuing this argument, we can always go up one level in both $C$ and $D$ and have $\varepsilon$ assign the next available integer to both these new elements. Similarly, we
work downwards starting from the original $2 + 2$. Thus, we obtain an isomorphism between $P$ and $U$. □

These lemmas lead us to the Embedding Theorem:

**Theorem 3.6.** If $P$ is a saturated order that does not contain $S$ as a suborder, then $P$ is isomorphic to a special suborder of $U^*$. Moreover, only the pivots of $P$ are labelled with non-integers.

**Proof.** This proof is similar to that of Lemma 3.5. After defining the labelling $\varepsilon$ on a starting configuration, we extend $\varepsilon$ in both directions. We then verify, step-by-step, that $\varepsilon$ defines an isomorphism.

The value $\varepsilon(v)$ is a non-integer iff $v$ is a pivot. We define $\varepsilon$ going upwards. If $\varepsilon(u)$ is already assigned and $uv$ is a vertical edge, then we assign $\varepsilon(v)$ uniquely by the rule $\lfloor \varepsilon(v) \rfloor = \lceil \varepsilon(u) \rceil + 1$. Thus, $\varepsilon$ is completely determined once a label has been assigned on each side.

If there is no suborder $1 + 4$, then we are done by Lemma 3.5. So the initial configuration is a $1 + 4$ suborder. We label the pivot by 2.5 and the 4 suborder
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by (0.5 or 1), 2, 3, (4 or 4.5). The numbering of the middle elements of the 4 is justified by Lemma 3.1.

Going upwards, the five possible cases are shown in Figure 7. (The mirror images are not shown.) Open circles represent elements of $P$ on which $\varepsilon$ defines an isomorphism. The elements to be considered next are solid circles and are shown with their forced labels. Recall that a non-integer label means a pivot. The first two cases begin with $2 + 2$ and no pivots at the top; the others begin with $1 + 4$ at the top—although we have not shown the lowest element of each 4 (which has the label 0.5 or 1).

For simplicity, we have shown definite values for $\varepsilon$ at each vertex (with the understanding that any fixed integer could be added to all values). The labelling determines the exact comparabilities that the new elements must satisfy. These new comparabilities are shown in the diagrams, but they must be proved. After the new elements are “added,” we have one of the five original cases, so that induction applies.

We now verify the new comparabilities in each case. The two sides $C$ and $D$ are on the left and right, respectively, of the diagrams. We write $\gamma$ and $\delta$ for the elements of $C$ and $D$, subscripted by the half-integer labels given in Figure 7.

a) By Lemma 3.4, the six elements form a suborder isomorphic to $H$.

b) By Lemma 3.4, $\{u, \gamma_4, \gamma_5, v, \delta_4, \delta_5\}$ is isomorphic to $H$. Since $\gamma_5$ is a pivot, it is incomparable with the three new elements on the right. The inequality $\gamma_4 < \delta_5$ prevents a $2 + 3$ suborder.

c) The inequality $\delta_2 < \gamma_4$ prevents a $2 + 3$ suborder. The vertical edge $e$ on the left is not in a $1 + 4$ because $\delta_3$ and $\delta_4$ (the only possible pivots) are not pivots. Therefore, $e$ is incomparable with an edge on the right and $\delta_3\delta_4$ is the only candidate.

d) The inequality $\delta_2 < \gamma_{4.5}$ prevents a $2 + 3$ suborder. Thus, $\gamma_{4.5}$ is incomparable with the top four elements on the right. The inequality $\gamma_{2.5} < \delta_5$ prevents $2 + 3$. The vertical edge $e$ on the left is not in a $1 + 4$ because $\delta_3$ and $\delta_4$ are not pivots. Therefore, $e$ is incomparable with $\delta_3\delta_4$.

e) By the labelling, the pivot $\gamma_{2.5}$ is the smallest element of the 4 associated with the pivot $\gamma_{4.5}$. Therefore, $\gamma_{4.5}$ is incomparable with the three new elements on the left. Both $\delta_2 < \gamma_4$ and $\delta_3 < \gamma_5$ hold to prevent any $2 + 3$. If $\delta_3 < \gamma_4$, then $S$ is a suborder of $P$, a contradiction.

We apply the dual argument when going downwards.

4. Proof of the first implication

In this section, we prove the implication (i)$\Rightarrow$(ii) of the Main Theorem.

We add $u$ and $v$ to $F_m(U)$ by requiring that $\alpha_0 < u < \alpha_1$ and $\beta_0 < v < \beta_1$. By the Crawley-Dean conditions, the new $m$-lattice is $F_m(U')$, where $U'$ is obtained from $U$ by “replacing” the edge $\alpha_0\alpha_1$ by $u$ and the edge $\beta_0\beta_1$ by $v$. (Of course, $u < x$ in $U'$ iff $\alpha_1 < x$ in $U$, etc.) Since $U$ and $U'$ are isomorphic, each element of $U$ is completely join- and meet-irreducible in $F_m(U)$. 

\[\square\]
For \( c \in U \), we write \( c^* \) and \( c_\ast \) for the unique upper and lower covers of \( c \) in \( F_m(U) \). When \( c = \alpha_n \), \( c^* = \alpha_n \lor \beta_{n-1} \). (The previous construction justifies this formula.) By duality and symmetry, this expression provides the formulas for \( c^* \) and \( c_\ast \), whenever \( c \in U \).

**Theorem 4.1.** If \( P \) is indecomposable slender order, then \( F_m(P) \) is isomorphic to an \( m \)-sublattice of \( F_m(U) \).

**Proof.** By Lemma 3.3, we can assume that \( P \) is saturated. We first assume that \( P \) does not contain \( S \). By Theorem 3.6, \( P \) is a special suborder of \( U^* \) in which only pivots have non-integer labels.

We shall define a mapping \( \varphi \) from \( P \) into \( F_m(U) \). Let \( n \) be an arbitrary integer. The letter \( k \) indicates an integer; values for each \( k \) will be specified later. We distinguish three cases:

- If \( P \) contains both \( \alpha_n \) and \( \beta_n \), then \( \varphi \alpha_n = \alpha_k \) and \( \varphi \beta_n = \beta_k \).
- If \( P \) contains \( \alpha_{n+1/2} \), then it also contains \( \beta_n \) and \( \beta_{n+1} \) by Lemma 3.1. In this case, \( \varphi \alpha_{n+1/2} = \alpha_k \), \( \varphi \beta_n = (\beta_k)_* \) and \( \varphi \beta_{n+1} = (\beta_k)^* \).
- If \( P \) contains \( \beta_{n+1/2} \), then it also contains \( \alpha_n \) and \( \alpha_{n+1} \) by Lemma 3.1. In this case, \( \varphi \beta_{n+1/2} = \beta_k \), \( \varphi \alpha_n = (\alpha_k)_* \) and \( \varphi \alpha_{n+1} = (\alpha_k)^* \).

Since \( P \) is a special suborder, an integer \( n \) can occur in at most one of the three cases above. Suppose that \( P \) contains \( \alpha_n \) but none of \( \beta_{n-1/2}, \beta_n \) or \( \beta_{n+1/2} \). Let \( e \) be the unique vertical edge that is incomparable with \( \alpha_n \). Since, \( e \) is not in any \( 2 + 2 \) or \( 1 + 4 \) suborder of \( P \), we have a contradiction. Therefore, if \( P \) contains \( \alpha_n \), then it contains \( \beta_{n-1/2}, \beta_n \) or \( \beta_{n+1/2} \) (and symmetrically). If some value of \( n \) does not occur in any of the three cases, then \( P \) is linearly decomposable, contrary to assumption. Therefore, \( \varphi \) is defined everywhere on \( P \) (once all values of \( k \) are specified).

We now assign the values of \( k \). For \( n = 0 \), let \( k = 0 \). All values of \( k \) are now uniquely determined by requiring that \( \varphi \) be isotone on \( P \cap A^* \) and that

\[
\{ k \mid \text{im}(\varphi) \text{ contains } \alpha_k \text{ or } \beta_k \} = \mathbb{Z}.
\]

It is immediate that \( \varphi \) defines an order-isomorphism from \( P \) into \( F_m(U) \). The \( m \)-sublattice \( F_m(U) \) generated by \( \varphi P \) is isomorphic to \( F_m(P) \).

We now assume that \( P \) contains suborders isomorphic to \( S \). Remove one comparability from each \( S \) to obtain \( R \) with the distinguished solid element shown in Figure 4. Perform the previous construction, in which each distinguished element will have the form \((\alpha_k)_* \) or \((\beta_k)_* \) for an integer \( k \). Replace each such \((\alpha_k)_* \) by its unique upper cover in \( F_m(U) - \{\alpha_k\} \). Use the analogous replacement for \((\beta_k)_* \). \( \Box \)

**5. From the order \( U \) to the order \( H \)**

In this section, we prove that \( F_m(U) \) is an \( m \)-sublattice of \( F_m(H) \), which proves the implication (ii) \( \Rightarrow \) (iii) of the Main Theorem.

Our description of \( F_m(H) \) in [5] is not needed in the next proof.

**Theorem 5.1.** \( F_m(U) \) is an \( m \)-sublattice of \( F_m(H) \).
Proof. Let $K$ be the suborder $\{\alpha_{-2}, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \beta_{-2}, \beta_{-1}, \beta_0, \beta_1, \beta_2\}$ of $U$. By removing doubly irreducibles from each side of $F_m(H)$, we obtain $F_m(K)$ as an $m$-sublattice of $F_m(H)$. (From each side of $F_m(H)$, remove the middle element of $H$; then remove one of the two new free generators.) An argument analogous to the one at the beginning of Section 4 justifies our conclusion.

We shall define the mapping $\varphi$ of $A$ into $F_m(K)$, omitting the symmetric case of $\varphi$ on $B$. By duality, it suffices to define $\varphi\alpha_n$ for $n \geq 0$. We begin by setting $\varphi\alpha_0 = \alpha_0$. Let $a = \alpha_1$ and $b = \beta_1$. Our inductive assumption is that $\varphi\alpha_n < a < \alpha_2$ and $\varphi\beta_n < b < \beta_2$ form $H$. After removing $a$ and $b$, we define $\varphi\alpha_n$ and $\varphi\beta_n$ as the original lower covers of $a$ and $b$, respectively. The new values of $a$ and $b$ are the upper covers of the original $a$ and $b$. Since $\varphi$ is an isomorphism from $U$ into $F_m(K)$, we conclude that $F_m(U)$ is an $m$-sublattice of $F_m(K)$. \hfill \Box

6. Breadth

In this section, we use attachments to show that $F_m(H)$ has breadth three which, together with Lemma 2.4, proves the implication $(iv) \Rightarrow (v)$ of the Main Theorem.

We start with our first basic concept from [6].

**Definition 6.1.** A family $A = \{A_i; \leq_i\}$ of orders is attachable to an order $P$, if the following two conditions are satisfied:

- **(Ord)** The relations $\leq_P$ and $\leq_i$ agree on $A_i \cap P$, for all $i \in I$.
- **(Int)** $A_i \cap A_j \subseteq P$, for $i \neq j$ in $I$.

Given such a family, we define the order

$$Q = \langle P \cup \bigcup_{i \in I} (A_i; \leq_i); \leq_Q \rangle$$

by defining $x \leq_Q y$ to hold, when one of the following five conditions holds:

- **(S)** $x, y \in P$ and $x \leq_P y$.
- **(A)** $x, y \in A_i$ and $x \leq_i y$, for some $i \in I$.
- **(AS)** $x \in A_i$, for some $i \in I$, $y \in P$ and there is $t \in A_i \cap P$ such that $x \leq_i t$ and $t \leq_P y$.
- **(SA)** $x \in P, y \in A_i$, for some $i \in I$, and there is $t \in A_i \cap P$ such that $x \leq_P t$ and $t \leq_i y$.
- **(AA)** $x \in A_i, y \in A_j$, for some $i \neq j \in I$, and there are $u, v \in P$ such that $x \leq_i u, u \leq_P v$, and $v \leq_j y$.

In Theorem 6.2, we state that $\leq_Q$ is an order relation. The order $\langle Q; \leq_Q \rangle$ is denoted by $P[A]$. We say that it is obtained by “attaching” the orders in $A$ to $P$.

The order $P$ is called the skeleton of $P[A]$ and each $A_i$ is called an attachment.

Here is the first main result of [6].

**Theorem 6.2.** Given an attachable family $A$ for an order $P$, the relation $\leq_Q$ is an order relation. Moreover, the restriction of $\leq_Q$ to $P$ equals $\leq_P$ and the restriction of $\leq_Q$ to $A_i$ equals $\leq_i$, for each $i \in I$.
Unfortunately, even if all orders in $\mathcal{A}$ are lattices and if $P$ is also a lattice, the order $P[A]$ is not a lattice in general. For this we need our second basic concept from [6].

**Definition 6.3.** A family $\mathcal{A} = \{(A_i; \lor_i, \land_i, 0_i, 1_i) \mid i \in I\}$ of nontrivial bounded lattices is lattice-attachable to a lattice $K$ when the following four conditions are satisfied:

(i) $A$ is attachable to $K$.
(ii) $K$ contains $0_i$ and $1_i$, for every $i \in I$.
(iii) $A_i \cap K$ is a sublattice of both $K$ and $A_i$, whenever $i \in I$.
(iv) The following two conditions are satisfied:

(C1) For $i \in I$ and $a \in A_i$, there is greatest element $\underline{a}$ of $(a)_{A_i} \cap K$ and a least element $\overline{a}$ of $[a]_{A_i} \cap K$.

(C2) Let $i \in I$ and $x \in K$. If $0_i \leq x$, then $[x]_K \cap A_i$ contains a greatest element $x^{(i)}$. If $x \leq 1_i$, then $[x]_K \cap A_i$ contains a smallest element $x^{(i)}$.

In [6], we proceed (Theorem 3 in [6]) to state necessary and sufficient conditions under which a lattice-attachable family for a lattice produces a lattice; when it does, we describe the joins and the meets. This result is very technical and luckily we only need the following sufficient conditions from [6]:

**Theorem 6.4.** Let $\mathcal{A} = (A_i \mid i \in I)$ be a lattice-attachable family for the lattice $K$. The order $L = K[A]$ is a lattice if the following two conditions are satisfied.

(a) For each $i \in I$, $A_i \cap K$ is a chain or equals $[0_i, 1_i]_K$.
(b) For $i \neq j \in I$, $A_i \cap A_j$ is a chain (possibly empty).

Now we can state our result on breadth.

**Theorem 6.5.** Let $C$ be a chain and let $\mathcal{A} = (A_i \mid i \in I)$ be a family of bounded lattices such $C \cap A_i$ is a convex subset of $C$ for each $i \in I$. If $\mathcal{A}$ is lattice-attachable to $C$, then the breadth of the lattice $L = C[A]$ is the maximum of the breadth of $A_i$ for $i \in I$.

**Proof.** By Theorem 6.4, $L$ is a lattice. Let $b_1, b_2, \ldots, b_n$, for $n \geq 3$, be distinct join-independent elements of $L$. We can assume that $b_1 \leq b_2 \leq \cdots \leq b_n$. Let $b_n \in A_j$ and write $B = A_j$. Suppose that $b_k \in A_i$ where $i \neq j$ and $1 < k < n$. Then

$$b_1 \leq b_k \lor_B b_n = b_k \lor_L b_n,$$

a contradiction. If $b_1$ is outside $B$, then $b_1 \lor_L b_k = b_1 \lor_B b_k$ whenever $k > 1$. In this case, $\{b_1, b_2, \ldots, b_n\}$ is a join-independent subset of $B$. Otherwise, all the original elements were in $B$. 

By the above proof, any join-independent set is almost completely inside one of the attachment lattices.

To prove the next result we make essential use of our description of $F_m(H)$. In fact, we adopt the notation of [5]. In particular, we change the meanings of $A$ and $B$: $A$ is now the countable lattice shown in Figure 4 of [5], reproduced here as Figure 8; $A$ and $B$ are isomorphic. The $m$-lattice $F_m(H) \cup \{0, 1\}$ consists of three
disjoint parts: the lattice $A$ on the left, $B$ on the right and a middle part $C$ which depends on $m$.

Let $I$ be the $[0, 1]$ interval of real numbers and let $R$ be the dyadic rationals in $I$. For uncountable $m$, the complete lattice $C$ is a linear sum over $I$, with $F_m(2 + 2) \cup \{0, 1\}$ at dyadic rationals and $2$ at all other values. Since $F_m(2 + 2)$ has breadth three, so does $C$. Following [5], $\{\alpha_r < \alpha'_r\} \cup \{\beta_r < \beta'_r\}$ is the $2 + 2$ at $r \in R$ and $\gamma_t < \gamma'_t$ is the $2$ at $t \in I - R$.

Our proof of Theorem 6.6 will apply Theorem 6.5 twice.

**Theorem 6.6.** $F_m(H)$ has breadth three.

**Proof.** Since $F(H)$ is a sublattice of every $F_m(H)$, we can assume that $m$ is uncountable. Since every finitely generated sublattice of $A$ is planar, $A$ has width two. Let $K$ be the complete chain which is the linear sum over $I$ with $\{\alpha_r, \alpha'_r\}$ at $r \in R$ and $\{\gamma_t, \gamma'_t\}$ at $t \in I - R$. Let $\overline{A}$ be the suborder $A \cup K$ of $F_m(H) \cup \{0, 1\}$. The complete lattice $\overline{A}$ has width two because it has dimension two. (The order $\overline{A}$ is a suborder of a direct product of two chains in which some points are substituted by 2-element chains.) Let $\overline{B} = B \cup L$, where $L$ is defined as for $K$ with $\beta$ replacing $\alpha$. The $m$-lattice $A \cup C$ is obtained by lattice-attaching the lattices $\overline{A}$ and $C$ over...
the skeleton $K$. By Theorem 6.5, $A \cup C$ has breadth three. We now lattice-attach $A \cup C$ and $B$ over the skeleton $L$ to form $F_m(H) \cup \{0,1\}$. The result has width three by Theorem 6.5. $\square$

**References**


