SUBDIRECTLY IRREDUCIBLE MODULAR LATTICES
OF WIDTH AT MOST 4

G. GRÄTZER AND H. LAKSER

Abstract. In 1970, R. Freese proved that the variety $M^4$ generated by modular lattices of width at most 4 has a finite basis. As an application, he obtained a complete description of all subdirectly irreducible members of this variety. We obtain an intuitive description of how congruences generated by a prime interval spread in a modular lattice of width at most 4, and apply the result to reprove Freese’s description of subdirectly irreducible lattices of width at most 4.

1. Introduction

Let $n$ be a natural number; a lattice $L$ is of width at most $n$, if every antichain in $L$ has $n$ or fewer elements. Let $M^n$ denote the variety generated by modular lattices of width at most $n$.

For $n \geq 5$, K. Baker [1] proved that $M^n$ is not finitely based. B. Jónsson observed that $M^3$ is finitely based.

The only open problem left, whether $M^4$ is finitely based, was decided in the classical work of R. Freese [4] (published in 1977; result earlier announced in 1970 and 1972; see [2] and [3]) by proving that there are exactly ten modular lattice varieties covering $M^4$ and any modular lattice varieties properly containing $M^4$, contains one of these ten varieties.

This method of proving that a modular lattice variety is finitely based was first employed in G. Grätzer [5], where it was proved that the variety generated by $M_3$ has exactly two covers generated by finite modular lattices; the condition on finiteness was removed in B. Jónsson [10], and the result was further generalized in D. X. Hong [9].

Freese’s main application of his result is a surprisingly easy and intuitive description of all subdirectly irreducible lattices of $M^4$—stated below as Freese’s Structure Theorem.

To state this result, we have to introduce snakes, an intuitive and visually easy to understand class of modular lattices of width at most 4. For a natural number $n$, we obtain an $n$-snake, $\text{Snake}(n)$, by gluing together $n$ copies of $M_3$: we start with $M_3$ and we glue a new $M_3$ alternatingly on the left and right top edge; Figure 1 shows a 3-snake and a 4-snake.

A snake can be extended to infinity upward, $\text{Snake}^{\uparrow}$, or downward, $\text{Snake}^{\downarrow}$, or in both directions, $\text{Snake}^{\leftrightarrow}$. If it is extended upward, $\text{Snake}^{\uparrow}$, we may add a unit element, and we get $S^+(\uparrow)$, or dually, $S^+(\downarrow)$. If it is extended upward and

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downward, Snake(↕), we may add a unit and/or a zero, so we get four variants: Snake(↕), S(↕), S(↕), S(↕), S(↕).

Let S be a snake. A level is an element x ∈ S that is both the unit of an M3, call it M, and the zero of an M3, call it M'. (In the finite case, Snake(n), a level could be defined as the set of elements of the same height i, with 2 ≤ i ≤ n − 2.) Define Λ(S) as the set of all levels of S. If x ∈ Λ(S) with M and M' as above, then exactly one atom of M is the zero of an M3 in S. Call the other two atoms of M free below x. Similarly, exactly one atom of M' is the unit of an M3; call the other two atoms of M' free above x. Then, for each x ∈ Λ, we get a four-element set
\[ \lambda(x) = \{ (b, b') \mid b \text{ is free below } x \text{ and } b' \text{ is free above } x \}. \]

Let ϕ be a function with domain (possibly empty) Λ' ⊆ Λ(S) such that ϕ(x) ∈ λ(x), for each x ∈ Λ'. To obtain the extended snake determined by ϕ, we add, for each x in the domain of ϕ, an extension, that is, an element dx,ϕ with b < dx,ϕ < b', where ϕ(x) = (b, b'). See Figure 2 for illustrations of extended snakes; the gray-filled elements are the extensions. Note that the 5-snake of Figure 2 is not planar.

We will use the notations Snake(n, ϕ), Snake(↑, ϕ), S+(↑, ϕ), and so on, in the obvious way.

Now we can state the structure theorem:

**Freese’s Structure Theorem.** The lattices

(i) \( C_2 \);
(ii) \( M_4 \);
(iii) \( \text{Snake}(n, \varphi) \);
(iv) \( \text{Snake}(↑, \varphi), S^+(↑, \varphi) \);
(v) \( \text{Snake}(↑, \varphi), S^+(↑, \varphi) \);
(vi) \( \text{Snake}(↑, \varphi), S^+(↑, \varphi), S^+(↑, \varphi) \),

are subdirectly irreducible modular lattices of width at most 4. And conversely, every nontrivial, subdirectly irreducible, modular lattice of width at most 4 occurs in this form.

The variety \( M^4 \) was the first “large” modular variety with a complete description of its subdirectly irreducible members. Now almost 40 years later, there is another one, described in G. Grätzer and D. Kelly [8].
Outline. In this paper, we follow the line of thought in [5] to obtain an intuitive description of how a congruence generated by a prime interval spreads in a subdirectly irreducible modular lattice of width at most 4.

If $K$ is a sublattice of a lattice $L$, we say that $K$ is a covering sublattice of $L$ if each prime interval in $K$ is prime in $L$.

Section 2 provides the reader with the background on projectivities in modular lattices; for more detail about the background, consult [7]. The new part of the paper starts on page 7; in Section 3, we show how strong a restriction it is that a modular lattice be of width at most 4 by showing that in such a lattice two covering $M_3$-s generate a very small sublattice. Some applications of this result are given in Section 4.

In Section 5, we introduce snakes and start looking at projectivities of prime intervals in width $\leq 4$ modular lattices. The intuitive concept of maximal snakes is introduced in Section 6, which concludes with the surprising statement that in a nondistributive, subdirectly irreducible, modular lattice of width $\leq 4$ that is not isomorphic to $M_4$, there is a unique maximal covering snake; see Theorem 21.

These results lay the groundwork for proving Freese's Structure Theorem in Section 7.

Notation. We use the notation of [7], available as a PDF file from http://www.math.umanitoba.ca/homepages/gratzer/notation.pdf

A number of small lattices appear again and again as sublattices in the discussion; their names and diagrams can be found in Figure 3.

When labeling the elements of $M_3$, such as $\{u,a,b,c,v\}$, the first element is always the zero, the last element the unit, and the others are the atoms. For a lattice $L$, let $A$ and $B$ be sublattices of $L$ isomorphic to $M_3$. We call $A$ upper adjacent to $B$, if an upper edge of $B$ is a lower edge of $A$. We define lower adjacent by interchanging “upper” and “lower”. Finally, $A$ and $B$ are adjacent, if $A$ is upper or lower adjacent to $B$. 

Figure 2. An extended 3-snake and an extended 5-snake.
Figure 3. Some important lattices.

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2. Projectivity of prime intervals, background

In this section, let $L$ be a modular lattice, let $p = [a, b], \ q = [c, d], \ r = [e, f]$ be intervals, and let $p \sim q \sim r$.

Lemma 1.

(i) Define the interval $q' = [c', d']$ by $c' = c \wedge (b \vee f)$ and $d' = b \vee f$. Then $p \sim q' \sim r$. The set $\{a, b, c', d', e, f\}$ generates in $L$ a sublattice as depicted in Figure 4 or a homomorphic image of the same.

(ii) If, in addition, $c' = a \vee e$, then the set $\{a, b, c', d', e, f\}$ generates in $L$ a distributive sublattice depicted as the first lattice in Figure 5 or a homomorphic image of the same.
(iii) Alternatively, if \( c' \neq a \lor e \), then the set \( \{ a \lor e \lor (b \land f), b \lor e, c' \lor (b \land f), a \lor f, d' \} \) in this lattice (and in all nondistributive homomorphic images) is a sublattice isomorphic to \( M_3 \).

(iv) If \( p \) is a prime interval, then so are \( q', r \), and \( p, q', \) and \( r \) either generate a homomorphic image of the first lattice of Figure 5 (the eight-element boolean lattice) or a nondistributive homomorphic image of the second lattice of Figure 5, that is, a homomorphic image not collapsing the \( M_3 \) sublattice.

For a proof, see Propositions 1 and 2 of [5]. For modular lattices of finite dimension, (iv) was first published in R. Thrall [11].

We need another result from [5] (Propositions 3 and 4), which we shall state only in the special case needed here.

Let \( p \) and \( r \) be prime intervals. If \( p \approx r \), then there exist a series \( \Sigma \) of prime intervals \( q_i = [c_i, d_i] \), for \( i = 0, \ldots, n \), connecting \( p \) and \( r \) (that is, satisfying \( p = q_0 \) and \( q_n = r \)) such that

1. \( p = q_0 \sim q_1 \sim q_2 \sim \cdots q_n = r \),
2. or dually
   \( p = q_0 \sim q_1 \sim q_2 \sim \cdots q_n = r \).

We shall call the series \( \Sigma \) normal, if whenever \( q_i \sim q_{i+1} \sim q_{i+2} = d_{i+1} \), and whenever \( q_i \sim q_{i+1} \sim q_{i+2} \), then \( c_i \land c_{i+2} = c_{i+1} \), for \( 0 \leq i < i + 2 \leq n \).

**Lemma 2.** If \( p \approx r \), then there exist a normal series \( \Sigma \) of intervals \( q_i = [c_i, d_i] \), for \( i = 0, \ldots, n \), connecting \( p \) and \( r \).

In particular, if \( L \) is a weakly atomic, subdirectly irreducible, modular lattice, then any two prime intervals of \( L \) can be connected with a normal sequence.

The following statement about minimal normal sequences easily follows:

**Lemma 3.** Let \( L \) be a weakly atomic, subdirectly irreducible, modular lattice. Let \( p \) and \( r \) be distinct prime intervals of \( L \), and let us choose a normal sequence \( \Sigma \) of prime intervals, \( q_i = [c_i, d_i] \), for \( i = 0, \ldots, n \), connecting \( p \) and \( r \) for which \( n \) is minimal. If \( n > 2 \), then

![Figure 4. Two step projectivity: general case.](image-url)
(i) the perspectivities $\sim^u$ and $\sim^d$ alternate in the minimal normal sequence $\Sigma$;
(ii) for $0 \leq i$ and $i + 2 \leq n$, whenever $q_i \sim^u q_{i+1} \sim^d q_{i+2}$ then $d_i \lor d_{i+2} = d_{i+1}$ and $c_i \lor c_{i+2} = c_{i+1}$, and dually, whenever $q_i \sim^d q_{i+1} \sim^u q_{i+2}$, then $c_i \land c_{i+2} = c_{i+1}$ and $d_i \land d_{i+2} > d_{i+1}$.

In other words, $q_i$, $q_{i+1}$, and $q_{i+2}$ generate a homomorphic image of the second lattice of Figure 5, or its dual, not collapsing the sublattice $M_3$.

Proof. Obviously, in a minimal normal sequence $\Sigma$, we cannot have two adjacent $\sim^u$ symbols or two adjacent $\sim^d$ symbols, proving (i).

Looking at the first lattice of Figure 5, we see that if $q_i \sim^u q_{i+1} \sim^d q_{i+2}$ with $d_i \lor d_{i+2} = d_{i+1}$ and $c_i \lor c_{i+2} = c_{i+1}$, then we can replace $q_{i+1}$ with $q_i^*$ so that $q_i \sim^d q_i^* \sim^u q_{i+2}$, not changing $n$, thereby contradicting the minimality of $n$ by (i).

We proceed similarly in the dual case, proving (ii). □

So a minimal normal sequence $\Sigma$ with $n > 2$, uniquely associates with each $0 < i < n$, a covering $M_3^i = \{0^i, m_1^i, m_2^i, m_3^i, 1^i\}$, where $q_i$ is either $[m_j^i, 1^i]$ (if $q_{i-1} \sim^u q_i \sim^d q_i^*$ or $[0^i, m_j^i]$ (if $q_{i-1} \sim^d q_i \sim^u q_i^*$), for some $1 \leq j_i \leq 3$.

We shall make use of the following immediate corollary of Lemma 3:

Lemma 4. Let $L$ be a weakly atomic, subdirectly irreducible, modular lattice. Let $p$ and $r$ be distinct prime intervals of $L$. Then either $p$ and $r$ generate a homomorphic image of the first lattice of Figure 5, or $p$ is either perspective up to a lower edge of a covering $M_3$ or perspective down to an upper edge of a covering $M_3$, and similarly for $r$. In particular, if $p \cap r \neq \emptyset$, then $p$ is either perspective up to a lower edge of a covering $M_3$ or perspective down to an upper edge of a covering $M_3$, and similarly for $r$.

Proof. Let $\Sigma$ be a normal sequence joining $p$ and $r$. If the length of the sequence, $n > 2$, then we get the desired $M_3$-s. If, on the other hand, $n = 2$, then either $p$ and $r$ generate a homomorphic image of the first lattice of Figure 5, or of the second lattice (or its dual), where the $M_3$ is not collapsed. This $M_3$ serves for both $p$ and $r$. Finally, if $n = 1$, then we have a perspectivity between $p$ and $r$, whereby they generate a homomorphic image of the first lattice of Figure 5.

Figure 5. Two step projectivity for prime intervals.
To establish the last statement of the lemma, we need only note that any homomorphism of the first lattice of Figure 5 that identifies \( a \) with \( e \) identifies \( b \) with \( f \), and dually, and that any homomorphism that identifies \( a \) with \( f \) or \( b \) with \( e \) collapses the whole lattice to a singleton. \( \square \)

3. Sublattices generated by two covering \( M_3 \)-s

In this section, we describe the sublattice generated by a pair of covering \( M_3 \)-s in a modular lattice \( L \) of width \( \leq 4 \).

**Theorem 5.** Let \( L \) be a modular lattice of width \( \leq 4 \) and let \( M, M' \) be distinct covering \( M_3 \)-s in \( L \). Then the sublattice of \( L \) generated by \( M \) and \( M' \) is either isomorphic to one of \( M_4 \) or \( M_3 \times C_2 \) or it is a homomorphic image collapsing no \( M_3 \) of the lattice \( M_3 + M_3 \) or \( M_{3,2,3} \).

**Note.** An equivalent way of describing the sublattice of \( L \) generated by \( M \) and \( M' \) is by stating that it is isomorphic to one of the lattices of Figure 3.

**Proof.** We first consider the case where the zeros of \( M \) and \( M' \) are identical, say \( u \). Since \( L \) is of width \( \leq 4 \), the element \( u \) has at most 4 covers in \( L \), so it follows that two atoms of \( M \) are each equal to an atom of \( M' \). Since \( M \neq M' \), the remaining atom of \( M \) is distinct from the remaining atom of \( M' \). Thus \( M \) and \( M' \) generate a lattice isomorphic to \( M_4 \).

We may henceforth assume that the zeros, and, by duality (since \( M_4 \) is selfdual), the units of \( M \) and \( M' \) are distinct.

The atoms of \( M \) and of \( M' \) form two three-element antichains, thus there are comparabilities among the atoms of \( M \) and \( M' \). There are then two cases to consider:

**Case 1:** Two distinct elements of one antichain are comparable to the same element of the other antichain. Let \( M = \{ u, a, b, c, v \} \) and \( M' = \{ u', a', b', c', v' \} \). Let \( a \leq a' \) and let \( c \) and \( c' \) be comparable.

Since \( a' \leq c \) implies the contradiction \( a \leq c \), therefore,

\[ c < a', \]

and so

\[ v = a \lor c \leq a'. \]

Since \( L \) is of width \( \leq 4 \), the set \( \{ a, b, c, b', c' \} \) is not an antichain, and thus there is comparability between one of \( a, b, c \) and one of \( b', c' \). By symmetry among \( a, b, c \) (all are \( \leq a' \)) and between \( b', c' \), we may assume that \( c \) is comparable to \( c' \). Now, \( c' \leq c \) implies the contradiction \( c' \leq c \leq a' \). Thus \( c < c' \) and so

\[ c \leq c' \land a' = u'. \]

Now \( u' \leq v \lor u' \leq a' \) and \( a' \) covers \( u' \). Thus, either \( v \lor u' = u' \), that is, \( v \leq u' \), and so we have a homomorphic image of \( M_3 + M_3 \) or \( v \lor u' = a' \) and so \( v \land u' = c \), and we have a homomorphic image of \( M_{3,2,3} \).

**Case 2:** No two distinct elements of either antichain are comparable to the same element of the other antichain.

We label the two \( M_3 \)-s as in Case 1, and we may assume, by width \( \leq 4 \), that \( a \leq a' \). The set \( \{ b, c, a', b', c' \} \) is not an antichain, and so one of \( b, c, \) must be comparable to one of \( a', b', c' \). Since \( a \leq a' \), neither \( b \) nor \( c \) is comparable to \( a' \).
By the symmetry between $b$ and $c$ and between $b'$ and $c'$, we may assume that $c$ is comparable to $c'$.

Since $a \leq a'$, the elements $c$ and $a'$ are incomparable. Thus $u \leq a' \land c < c$. Since $c$ covers $u$, we conclude that

$$a' \land c = u.$$  

We now consider the two subcases:

Case 2a: $c' \leq c$. Then $u' = a' \land c' \leq a' \land c = u$, and so

$$u' \leq u < a \leq a'.$$

But $a'$ covers $u'$. Consequently, $u' = u$, contradicting our assumption that $M$ and $M'$ have distinct zeros. Thus this subcase cannot occur.

Case 2b: $c < c'$. Then

$$v = a \lor c \leq a' \lor c' = v'$$

and

$$u = a \land c \leq a' \land c' = u'.$$

Now, $c \not\leq u'$. Thus

$$c \lor u' = c'$$

and $c \land u' = u$

since $c'$ covers $u'$ and $c$ covers $u$. Similarly,

$$a \lor u' = a'$$

and $a \land u' = u$

since $a \not\leq u'$,

$$v \land a' = a$$

and $v \lor a' = v'$

since $v \not\leq a'$, and

$$v \land c' = c$$

and $v \lor c' = v'$

since $v \not\leq c'$. Furthermore,

$$v \lor u' = a \lor c \lor u' = a' \lor c' = v'.$$

Now, $b \not\leq u'$ since $b$ is comparable to at most one of $a'$, $b'$, $c'$. Thus $b \lor u'$ covers $u'$, and, clearly, $b \lor u' \leq v'$. Since $a$ is comparable to $a'$ and $c$ is comparable to $c'$, we conclude that

$$b \lor u' \neq a', c'.$$

Thus, either $b \lor u' = b'$, and so we have $M_3 \times C_2$, or $b \lor u'$ is a new atom $d$ of the interval $[u', v']$. But, since $v \lor u' = v' \neq v$, then $\{v, a', b', c', d\}$ would be a five-element antichain, a contradiction.

\[\square\]

4. Some applications

We begin with the following observation:

Lemma 6. A subdirectly irreducible modular lattice $L$ of width $\leq 4$ is weakly atomic. Therefore, the base congruence of $L$ is $\text{con}(p)$, for any prime interval $p$ of $L$.

Proof. Indeed, if $L$ is distributive, then $|L| \leq 2$, and the statement is trivial. If $L$ is not distributive, then it contains a sublattice $M = \{u, a, b, c, v\} \cong M_3$. We claim that $[u, a]$ contains a prime interval. Indeed, otherwise it would contain $C_5$ as a sublattice, so $L$ would contain $C_5^2$ as a sublattice, which is of width 5. Therefore, $L$ has prime intervals. Let $p$ be a prime interval of $L$. Then $\text{con}(p)$ is an atom of the congruence lattice and it is contained in every nontrivial congruence, hence, it is the base congruence. \[\square\]
Lemma 7. Let $L$ be nondistributive, subdirectly irreducible, modular lattice of width at most 4. Then:

(i) any prime interval in $L$ is either perspective down to an upper edge of a covering $M_3$ or perspective up to a lower edge of a covering $M_3$;

(ii) $L$ contains a covering $M_3$.

Proof. By Lemma 6, there is a prime interval $p = [x, y]$ in $L$. Since $L$ is not distributive, it is of length $\geq 2$. Thus, by duality, we may assume that there is $z \in L$ with $y < z$. By Lemma 6, $[y, z]$ contains a prime interval $r$. But the sublattice generated by $p$ and $r$ is not a homomorphic image of the first lattice of Figure 5. Thus, by Lemma 4, $p$ is either perspective up to a lower edge of a covering $M_3$ or perspective down to an upper edge of a covering $M_3$, establishing (i).

Statement (ii) follows immediately from (i), since $L$, being weakly atomic (and not a singleton), contains a prime interval. □

We next consider two special consequences of dealing with lattices of width at most 4.

Lemma 8. Let $L$ be a lattice of width $\leq 4$. Then $L$ cannot contain the configuration of Figure 6 consisting of two $M_3$’s and a $C_2^2$, where the dotted edges may be collapsed, and satisfying $[z, u] \sim [p, q]$ and $[p, r] \sim [a, d]$.

![Figure 6. Excluded configuration.](image)

Proof. Consider the set $\{x, y, p, b, c\}$. Since $L$ has width $\leq 4$, this set is not an antichain. Now $\{x, y, p\}$ is an antichain since $u \land p = z$, and $\{p, b, c\}$ is an antichain since $p \land d = a$. Thus one of $x, y$ is comparable to one of $b, c$. By symmetry, we may then assume that $y \leq b$.

Now $y \leq b$ implies that $y \leq r$, which, in turn, implies the contradiction $q = u \lor p = y \lor z \lor p = y \lor p \leq r$. □

We also need the following statement.

Lemma 9. Let $L$ be a modular lattice of width $\leq 4$. Let $[p, q]$ and $[p, r]$ be distinct prime intervals in $L$. Let $[p, q] \sim [t, x]$, a lower edge of a covering $M_3$, and let $[p, r] \sim [a, d]$, an upper edge of a covering $M_3$, as in Figure 7. Set $s = q \lor r$. If the interval $[p, s]$ does not contain an $M_3$, then $[r, s] \sim [t, x]$. 
Proof. We refer to Figure 7. The lattice $L$ is of width $\leq 4$, so the set $\{x, y, z, b, c\}$ is not an antichain, while $\{x, y, z\}$ and $\{b, c\}$ are. Thus there is comparability between one of $b, c$ and one of $x, y, z$. Since $a \leq p \leq t \leq x, y, z$, if one of $x, y, z$ were $\leq$ one of $b, c$, we would get the contradiction $a \leq b$ or $a \leq c$.

Thus one of $b, c$ is $\leq$ one of $x, y, z$. By symmetry between $b$ and $c$ and between $y$ and $z$, we may assume that either $c \leq x$ or $c \leq z$. We show that, in fact, $c \leq x$.

Assume, to the contrary, that $c \not\leq x$. Then $c \leq z$; it follows that $r = p \lor d = p \lor a \lor c = p \lor c \leq z$.

Thus, using $r \leq z$, we obtain that

$$s \lor t = q \lor r \lor t = x \lor r \leq x \lor z = u.$$

Now $r \not\leq x$, since $c \not\leq x$. Thus $x \lor r > x$, that is, $x \lor r = u$. We conclude that

$$s \lor t = u.$$

From $[p, q] \nleq [t, x]$, we conclude that $q \not\leq t$. Also, $r \not\leq t$, because $r \leq t$ would imply that $c \leq x$, contradicting the assumption that $c \not\leq x$. Since the interval $[p, s]$ contains no $M_3$, it now follows that

$$s \land t = p.$$

By modularity, the two last displayed equations imply that the intervals $[t, u]$ and $[p, s]$ are isomorphic, contradicting that one contains an $M_3$ and the other one does not. This contradiction thus shows that $c \leq x$.

Then $s = q \lor c \leq x$, and, since $s \lor t \geq q \lor t = x$, we conclude that

$$s \lor t = x.$$

We now show that $s \land t = r$. Since $s \not\leq t$, it follows that $p \leq s \land t < s$. Furthermore, $s \land t \neq q$ since $q \not\leq t$. The interval $[p, s]$ contains no $M_3$. Thus either $s \land t = p$ or $s \land t = r$. Since $x \succeq t$, it follows by modularity, that $s \succeq s \land t$, so

$$s \land t = r.$$
Thus \([r, s] \cup [t, x]\), as claimed.

Now we are ready to state and prove the special form of Theorem 5 for subdirectly irreducible lattices.

**Theorem 10.** Let \(L\) be a subdirectly irreducible modular lattice of width \(\leq 4\), and let \(M\) and \(M'\) be distinct covering \(M_3\)-s in \(L\). Then the sublattice generated by \(M\) and \(M'\) is isomorphic to either the lattice \(M_3 + M_3\) or the lattice \(M_4\) or it is a homomorphic image of the lattice \(M_3 + M_3\), collapsing no \(M_3\).

**Proof.** By Theorem 5, we need only show that \(L\) contains no sublattice isomorphic to the lattice \(M_3 \times C_2\) covering \(M_3\)-s in \(L\).

First, assume that \(L\) contains a sublattice isomorphic to \(M_3 \times C_2\), as in Figure 8, that is, \(a < u\). By Lemma 6, the interval \(\langle a, u \rangle\) contains a prime interval \([p, q]\). (All the solid lines in the diagram are coverings.) Set \(r = p \lor d\) and \(s = q \lor r\). Then \([p, r]\) is a prime interval.

![Figure 8. A gap.](image-url)
cannot be perspective up to a lower edge of an $M_3$. We thus have a violation of the last statement Lemma 4.

\section{Snakes and Projectivities}

We first give a formal definition of a snake in a lattice.

Recall that an order $S$ is of locally fine length, if each bounded interval $[x, y]$ in $S$ has finite length, that is, there is a finite bound on the length of the chains in $[x, y]$.

**Definition.** A snake is an order $S$ of locally finite length that is a union of a nonempty set $M$ of $M_3$-s satisfying the following two conditions:

(i) Given two distinct elements of $M$, either the unit of one is $\leq$ the zero of the other, or else they are adjacent with their union isomorphic to $M_3$.

(ii) If $M \in M$ and its unit is not a maximal element of $S$, then it is lower adjacent to exactly one element of $M$, and, dually, if its zero is not a minimal element of $S$, then it is upper adjacent to exactly one element of $M$.

We note that if $M = \{u, a, b, c, v\}$ is an element of $M$ with $u$ not minimal in $S$ and $v$ not maximal in $S$, and if $[u, a]$ is an upper edge of $M' \in M$, then the upper edge of $M$ that is a lower edge of an $M'' \in M$ is $[a, v]$—otherwise $M'$ and $M''$ violate Condition (i).

It is easy to see that locally a snake is a gluing of the $M_3$-s in the defining $M$. It is then clear that a snake is a locally finite modular lattice, indeed simple and of width $\leq 4$ (of exact width 4 if there is more than one $M_3$). It is also easy to see that $M$ consists of all the $M_3$-s in $S$. A snake can be infinite. If the snake consists of $n$ copies of $M_3$-s, we say that it is an $n$-snake—see Figure 1 for two small examples. Obviously, an $n$-snake is unique up to isomorphism.

We first determine the prime intervals in a snake.

**Lemma 11.** Let $M$ be a nonempty set of $M_3$-s such that $S = \bigcup M$ is a snake. Then any prime interval $[x, y]$ in $S$ is a prime interval in some $K \in M$.

**Proof.** There are three cases to consider.

**Case 1:** $x$ is the zero of some $K \in M$. Then we show that $y$ is one of the atoms of $K$, proving our claim. For, otherwise, there is a $K_1 \in M$ distinct from $K$ with
y ∈ K_1. Then u_1, the zero of K_1, cannot be ≥ the unit of K. Thus, by Condition (i) of the Definition, K_1 is upper adjacent to K, that is, u_1 is an atom of K. Then y ≥ u_1 > x, and so y = u_1, contradicting our assumption that y is not an atom of K. Thus our claim holds in this case.

Case 2: x is an atom of some K ∈ M. If y = v, the unit of K, we are done. Otherwise, y ∈ v and there is a K_1 ∈ M distinct from K with y ∈ K_1. Then, by Condition (i) of the Definition, K_1 is upper adjacent to K and K_1 ∪ K ≅ M_{3,3}. Since y ∈ v, x must be the atom of K that is the zero of K_1. That is, x is the zero of K_1, and we are back in Case 1.

Case 3: x is the unit of some K ∈ M. Then x is not maximal in S and so, by Condition (ii) of the Definition, there is a K_1 ∈ M upper adjacent to K. But then x is an atom of K_1 and we are back in Case 2.

Thus the claim of the lemma holds.

We say the snake S in a lattice L is a covering snake in L if S is a covering sublattice of L. By Lemma 11, we have:

**Lemma 12.** Let M be a nonempty set of M_3-s in a lattice L, and let S = ∪M be a snake. Then S is a covering snake in L iff each element of M is a covering M_3 in L.

We now characterize covering snakes in a subdirectly irreducible modular lattice of width ≤ 4.

**Lemma 13.** Let L be a subdirectly irreducible modular lattice of width ≤ 4, and let the sublattice S of locally finite length be the union of a set M of covering M_3-s in L. Then S is a covering snake in L iff the following two conditions hold:

(i') The units (equivalently, the zeros) of distinct elements of M are distinct.

(ii') If K ∈ M, and its unit v is not the maximal element of S, then there is a K' ∈ M upper adjacent to K; and dually.

Proof. Clearly, if S is a snake, then Conditions (i') and (ii') of our lemma hold.

Now, let Conditions (i') and (ii') of our lemma hold, and refer to Theorem 10. By Condition (i'), two distinct elements of M generate a homomorphic image of M_3 + M_3 or of M_{3,3}, that is, they satisfy Condition (i) of the Definition. Furthermore, by (the dual of) Lemma 8, two distinct upper edges of K cannot be lower edges of M_3-s. Thus Condition (ii') of our lemma is equivalent to Condition (ii) of the Definition. Consequently, S is a snake.

**Lemma 14.** Let L be a subdirectly irreducible modular lattice of width ≤ 4, and let K_0, ..., K_{n-1} be a sequence of covering M_3-s in L with the property that if K_i ≠ K_{i+1}, then K_i and K_{i+1} are adjacent, for each i with 0 ≤ i < n − 1. Then S = ∪(K_i | 0 ≤ i < n) is a finite covering snake in L.

Proof. We apply Lemma 13. Clearly, S is locally finite.

We first establish Condition (i') of Lemma 13 for the set \{ K_i | 0 ≤ i < n \}. Assume, to the contrary, that K_i and K_j are distinct and have either a common zero or a common unit. Then, by Theorem 10, K_i ∪ K_j is a sublattice isomorphic to M_4. Now, one of i, j, say i, is distinct from n − 1, and so K_i and K_{i+1} generate a sublattice isomorphic to M_{3,3}. Thus the sublattice generated by K_i, K_{i+1}, and K_j is as in Figure 10, or its dual, which contains a five-element antichain (the black-filled elements), thereby contradicting the hypothesis of width ≤ 4. Consequently, Condition (i') holds.
We next establish Condition (ii'). For each \( i \), we denote by \( v_i \) the unit of \( K_i \). By Theorem 10, \( C = \{ v_i \mid 0 \leq i < n \} \) is a chain. Let \( v_i \) be not maximal in \( S \). Then \( v_i \) is not maximal in \( C \). Let \( j \) be the largest \( m \) with \( 0 \leq m < n \) and \( v_m > v_i \). We have two cases:

**Case 1:** \( j < n - 1 \). Then \( v_j > v_i \) and \( v_{j+1} \leq v_i \), so \( K_j \) and \( K_{j+1} \) are distinct. By our hypothesis concerning \( K_j \) and \( K_{j+1} \), either \( v_j < v_{j+1} \) or \( v_{j+1} < v_j \). Thus \( v_{j+1} \prec v_j \) and so \( v_{j+1} = v_i \). By Condition (i') proved above, \( K_i = K_{j+1} \). Then, since \( v_j > v_{j+1} \), a lower edge of \( K_j \) is an upper edge of \( K_i = K_{j+1} \), establishing Condition (ii') in this case.

**Case 2:** \( j = n - 1 \), that is, \( v_{n-1} > v_i \). Let \( k \) be the largest \( m \), \( 0 \leq m < n \) with \( v_m \leq v_i \). Then \( i < k < n - 1 \) and \( v_{k+1} > v_i \geq v_k \). But \( v_{k+1} > v_k \). Thus \( v_i = v_k \). Then \( K_i = K_k \) and so \( K_{k+1} \) is upper adjacent to \( K_i \), establishing Condition (ii') in this case also. □

We can now present the characterization of projectivity of prime intervals.

**Theorem 15.** Let \( L \) be a subdirectly irreducible modular lattice of width \( \leq 4 \) and let \( p \) and \( r \) be distinct prime intervals of \( L \). Then either \( p \) and \( r \) generate a homomorphic image of the first lattice of Figure 5, or there is a finite covering snake \( S \) in \( L \) where \( p \) is either perspective up to a lower interval or perspective down to an upper interval of an \( M_3 \) in \( S \) and \( r \) is either perspective up to a lower interval or perspective down to an upper interval of an \( M_3 \) in \( S \).

**Proof.** Assume that \( p \) and \( r \) do not generate a homomorphic image of the first lattice of Figure 5. Then, by Lemma 3, there is a normal sequence \( \langle q_i = [c_i, d_i] \mid 0 \leq i \leq n \rangle \) of minimal length joining \( p \) and \( r \). This normal sequence yields a sequence \( \langle K_i \mid 0 \leq i < n - 1 \rangle \) of covering \( M_3 \)-s: if \( q_i \overset{u}{\sim} q_{i+1} \overset{d}{\sim} q_{i+2} \), then

\[
K_i = \{ c_i \lor c_{i+2}, d_i \or c_{i+2}, c_i \lor d_{i+2}, d_i \lor d_{i+2} \},
\]

and if \( q_i \overset{d}{\sim} q_{i+1} \overset{u}{\sim} q_{i+2} \), then

\[
K_i = \{ c_{i+1}, c_i \land d_{i+2}, d_{i+1}, d_i \land c_{i+2}, d_i \land d_{i+2} \}.
\]

We claim that \( S = \bigcup \langle K_i \mid 0 \leq i < n - 1 \rangle \) is the desired covering snake—we need only show that \( S \) is a snake. We apply Lemma 14. Let \( K_i \neq K_{i+1} \). By duality, we may assume that

\[
K_i = \{ c_i \lor c_{i+2}, d_i \lor c_{i+2}, c_i \lor d_{i+2}, d_i \lor d_{i+2} \}.
\]

Then

\[
K_{i+1} = \{ c_{i+2}, c_{i+1} \land d_{i+3}, d_{i+2}, d_{i+1} \land c_{i+3}, d_{i+1} \land d_{i+3} \}.
\]
Then, by Theorem 10, $K_i$ and $K_{i+1}$ generate either $M_4$ or $M_{3,3}$. We show that they cannot generate $M_4$, that is, that we cannot have both their zeros equal and their units equal. Assume, to the contrary, that $c_i \lor c_{i+2} = c_{i+2}$ and $d_{i+1} \land d_{i+3} = d_{i+1}$, that is, $c_i \leq c_{i+2}$ and $d_{i+1} \leq d_{i+3}$. Then

$$K_i = \{c_{i+2}, c_{i+1}, d_{i+2}, d_i \lor c_{i+2}, d_{i+1}\}$$

and

$$K_{i+1} = \{c_{i+2}, c_{i+1}, d_{i+2}, d_{i+1} \land c_{i+3}, d_{i+1}\}.$$ 

(See Figure 11.) But then $d_i \lor c_{i+2}$ and $d_{i+1} \land c_{i+3}$ are distinct atoms of the $M_4$ and, so,

$$[c_{i+2}, d_i \lor c_{i+2}] \sim [d_{i+1} \land c_{i+3}, d_{i+1}].$$ 

Since $q_i = [c_i, d_i] \sim [c_{i+2}, d_i \lor c_{i+2}]$ and $[d_{i+1} \land c_{i+3}, d_{i+1}] \sim [c_{i+3}, d_{i+3}] = q_{i+3}$, we get $q_i \sim q_{i+3}$, contracting the minimality of the normal sequence.

Consequently, if $K_i$ and $K_{i+1}$ are distinct, they generate $M_{3,3}$. Thus, by Lemma 14, $S$ is a snake, concluding the proof.

\□

6. Maximal snakes

It is immediate from Zorn’s lemma, that, in any lattice, a snake can be extended to a maximal snake, and that a covering snake can be extended to a maximal covering snake. By a \textit{maximal covering snake} we mean a covering snake that is not properly contained in any other covering snake.
We show here, without using Zorn’s Lemma, that, with one exception, a nondistributive subdirectly irreducible modular lattice $L$ of width $\leq 4$ contains a unique maximal covering snake, the union of all the covering $M_3$-s in $L$. The one exception is the lattice $M_4$. We first dispose of this case.

First, a preliminary lemma:

**Lemma 16.** Let $L$ be a subdirectly irreducible modular lattice of width $\leq 4$, and let $M = \{u, a, b, c, v\}$ be a covering $M_3$ in $L$. If the edge $[a, v]$ is perspective up to a lower edge of a sublattice $K \cong M_3$, then $[a, v]$ is that lower edge of $K$. If $[a, v]$ is perspective down to an upper edge distinct from $[a, v]$ of a sublattice $K \cong M_3$, then this upper edge of $K$ is a lower edge of $M$.

*Proof.* This follows immediately from Theorem 10, for such $K$ and $M$ cannot generate $M_4$ or a homomorphic image of $M_3 + M_3$. □

We now dispose of $M_4$:

**Lemma 17.** Let $L$ be a subdirectly irreducible modular lattice of width $\leq 4$. If $L$ contains a covering sublattice $M$ isomorphic to $M_4$, then $L = M$.

*Proof.* Set $M = \{u, a, b, c, d, v\}$ with $u < a, b, c, d < v$. We show that $L$ can contain no other elements. Assume, to the contrary, that $L$ contains an additional element. By width $\leq 4$, this element cannot be in the interval $[u, v]$. Thus $L$ contains an element $z$ such that $z > v$ or $z < v$; by duality, we may assume that $z > v$. By Lemma 6, there is a prime interval $[x, y]$ in the interval $[v, z]$. We apply Theorem 15 to the prime intervals $[a, v]$ and $[x, y]$. Since $v \leq x$, the intervals $[x, y]$ and $[a, v]$ do not generate a homomorphic image of the first lattice of Figure 5. There is thus a finite covering snake $S$ in $L$ and there are covering $M_3$-s in $S$, $K_1$ and $K_2$, such that $[a, v]$ is either perspective up to a lower edge or perspective down to an upper edge of $K_1$ and $[x, y]$ is either perspective up to a lower edge or perspective down to an upper edge of $K_2$. Note that the unit of $K_2$ cannot be $\leq v$.

By virtue of the excluded configuration Figure 10, $[a, v]$ cannot be a lower edge of $K_1$, nor can any lower edge of $M$ be an upper edge of $K_1$. Thus, by Lemma 16, $K_1$ is one of the three $M_3$-s in $M$ that contain $a$, say $K_1 = \{u, a, b, c, v\}$. Thus $\{u, a, b, c, v\}$ is an $M_3$ in $S$. Since $v$ is not $\geq$ the unit of $K_2$, the element $v$ is not maximal in $S$. Consequently, one of $[a, v], [b, v], [c, v]$ is a lower edge of some $M_3$ in $S$, yielding the excluded configuration of Figure 10. This contradiction establishes the lemma. □

We now can restrict our attention to subdirectly irreducible lattices of width $\leq 4$ that are not isomorphic to $M_4$. As an immediate consequence of the above lemma we have:

**Corollary 18.** Let $L$ be a subdirectly irreducible modular lattice of width $\leq 4$ that is not isomorphic to $M_4$. Then any two distinct covering $M_3$-s in $L$ generate a sublattice that is either a homomorphic image of $M_3 + M_3$ or isomorphic to $M_{3,3}$.

Now we are getting close to establishing the uniqueness of the maximal covering snake.

**Lemma 19.** Let $L$ be a subdirectly irreducible modular lattice of width $\leq 4$ that is not isomorphic to $M_4$. Let $S$ be a covering snake in $L$ and let $K$ be a covering $M_3$ in $L$ adjacent to a covering $M_3$ in $S$. Then $S' = S \cup K$ is a covering snake in $L$. 

Proof. If $K$ is in $S$ we are done. So, assume $K$ is not in $S$.

By duality, we may assume that $K$ is upper adjacent to a covering $K_1 \cong M_3$, with unit $v_1$, in $S$. Then $v_1$ is maximal in $S$. For, otherwise, there would be a covering $K_2 \cong M_3$ in $S$ also upper adjacent to $K_1$, and then $K$ and $K_2$ would violate Lemma 18. Thus $S'$ is just $S$ with a new $M_3$ added upper adjacent to the top $M_3$ of $S$.

That $S'$ is a snake follows immediately from Lemma 13. $S'$ is of locally finite length because we have added only finitely many elements to $S$.

By Lemma 18, Condition (i') of Lemma 13 holds.

Condition (ii') and its dual also hold. The unit of $K$ is maximal in $S'$, and the only covering $M_3$ in $S$ whose unit is maximal in $S$ is $K_1$, and $K$ is upper adjacent to $K_1$, that is, Condition (ii') holds. If the zero $u$ of a covering $K_2 \cong M_3$ in $S$ is not minimal in $S'$, then $u$ is not minimal in $S$, and so $K_2$ is upper adjacent to an $M_3$ in $S$, establishing Condition (ii').

Thus $S'$ is a snake. □

We actually only need Lemma 19 for finite covering snakes.

Lemma 20. Let $L$ be a nondistributive, subdirectly irreducible, modular lattice of width $\leq 4$ that is not isomorphic to $M_4$, and let $K_1$ and $K_2$ be covering $M_3$-s in $L$. Then there is a finite covering snake $S$ in $L$ that contains both $K_1$ and $K_2$.

Proof. If $K_1 = K_2$, then $K_1$ is the desired snake.

If $K_1$ and $K_2$ generate $M_3$, then $K_1 \cup K_2$ is the desired snake.

Otherwise, by Lemma 18, we may assume that the unit of $K_1$ is $\leq$ the zero of $K_2$. Let $p$ be an upper edge of $K_1$ and $r$ a lower edge of $K_2$. $p$ and $r$ do not generate a homomorphic image of the first lattice of Figure 5. Thus, by Theorem 15, there is a finite covering snake $S$ in $L$, a covering $M_3$, denoted $K_3$, in $S$ with $p$ either perspective up to a lower edge of $K_3$ or perspective down to an upper edge of $K_3$, and a covering $M_3$, denoted $K_4$, in $S$ with $r$ either perspective up to a lower edge of $K_4$ or perspective down to an upper edge of $K_4$. By Lemma 16, we conclude that $K_1$ and $K_2$ are adjacent. Thus, by Lemma 19, $S \cup K_1$ is a covering snake, and, since $K_2$ and $K_3$ similarly are adjacent, $S \cup K_1 \cup K_2$ is then the desired finite covering snake. □

Theorem 21. Let $L$ be a nondistributive, subdirectly irreducible, modular lattice of width $\leq 4$ that is not isomorphic to $M_4$. Then $L$ has a unique maximal covering snake, the union of all covering $M_3$ sublattices.

Proof. Let $\mathcal{M}$ be the set of all covering $M_3$-s in $L$. By Lemma 7, $\mathcal{M}$ is not empty. Set $S = \bigcup \mathcal{M}$. Then, clearly, any covering snake in $L$ is a subset of $S$. We thus need only show that $S$ is a snake. We use Lemma 13.

By Lemma 18, Condition (i') of Lemma 13 holds.

Next, let $K_1 \in \mathcal{M}$ with unit $v_1$ not maximal in $S$. Then there is $K_2 \in \mathcal{M}$ with unit $v_2 > v_1$. By Lemma 20, there is a (finite) covering snake $S'$ in $L$ containing both $K_1$ and $K_2$. Since $v_1$ is not maximal in $S'$, there is a covering $M_3$ in $S' \subseteq S$ upper adjacent to $K_1$, establishing Condition (ii') of Lemma 13.

Finally, $S$ is of locally finite length. Indeed if $x < y$ are elements of $S$, then they are elements respectively of covering $M_3$-s $K_1$ and $K_2$ and so, as above, there is a finite covering snake $S'$ in $L$ containing both $x$ and $y$. Let $x = x_0 < x_1 < \cdots < x_n = y$ be a maximal chain in $S'$. Then, since $S'$ is a covering sublattice of $L$, it is
a maximal chain in \( L \), and so, *a fortiori*, in \( S \). By modularity, \( S \) is of locally finite length.

Thus, \( S \) is a covering snake, and the unique maximal one. \( \square \)

### 7. The proof of Freese's Structure Theorem

A snake is a simple modular lattice of width \( \leq 4 \). Any prime interval in one of the lattices \( \text{Snake}(n, \varphi) \), \( \text{Snake}(\uparrow, \varphi) \), \( \text{Snake}([, \varphi) \), and \( \text{Snake}([, \varphi) \) is perspective to a prime interval in the underlying snake. Thus all of these extended snakes are also simple, and they are clearly modular and of width to a prime interval in the underlying snake. Thus all of these extended snakes are also simple, and they are clearly modular and of width \( \leq 4 \). The lattices \( S^+(\downarrow, \varphi), S_+([, \varphi), S^+([, \varphi), S_+([, \varphi), \) and \( S^+(\downarrow, \varphi) \) are then subdirectly irreducible (nonsimple) modular lattices of width \( \leq 4 \).

To complete the proof of the Structure Theorem we need only consider a non-distributive, subdirectly irreducible, modular lattice \( L \) of width \( \leq 4 \) that is not isomorphic to \( M_4 \). By Theorem 21, \( L \) has a unique maximal covering snake \( S \), consisting of all the covering \( M_3 \)-s in \( L \).

**Lemma 22.** Any prime interval \( p \) in \( L \) is either perspective up to a lower edge of a covering \( M_3 \) in \( S \) or perspective down to an upper edge of a covering \( M_3 \) in \( S \).

**Proof.** This follows immediately from Lemma 7. \( \square \)

**Lemma 23.** \( S \subseteq L \) has at most one upper bound and, dually, at most one lower bound.

**Proof.** Assume, to the contrary, that \( S \) has two distinct upper bounds. Then \( S \) has upper bounds \( x \) and \( y \) with \( x < y \). Then, by Lemma 6, there is a prime interval \( p \subseteq [x, y] \). Since \( x \) is an upper bound of \( S \), \( p \) can neither be perspective up nor perspective down to an interval in \( S \). This contradiction to Lemma 22, and the dual argument, establish our lemma. \( \square \)

We now look at covers (in \( L \)) of elements of \( S \).

**Lemma 24.** Let \( K = \{ u, a, b, c, v \} \) be a covering \( M_3 \) in \( S \). The only covers in \( L \) of \( u \) are \( a \), \( b \), and \( c \).

**Proof.** Assume, to the contrary, that \( u \) has a cover \( x \) distinct from \( a \), \( b \), \( c \). Then \( x \nleq v \), for otherwise, \( K \cup \{ x \} \) would be a covering \( M_4 \). Then, since \( S \) has at most one upper bound, \( v \) is not maximal in \( S \). There is then a covering sublattice \( K_1 \cong M_3 \) in \( S \) that has a lower edge that is an upper edge, say \( \{ a, v \} \), of \( K \). So \( K_1 = \{ u_1 = a, a_1, b_1, c_1 = v, v_1 \} \). Let us count height in \( L \) up from \( u \). Then \( x \lor b, x \lor c, a_1, b_1, c_1 = v \) are all of height 2, and so any two of them are either equal or incomparable. They cannot be all distinct because they would form an antichain of 5 elements. Thus two of them must be equal. But \( a_1, b_1, v \) are distinct. Consider \( x \lor b \). Since \( b \nleq a_1, b_1 \), it follows that \( x \lor b \) is distinct from \( a_1 \) and \( b_1 \). But \( x \nleq v \), so \( x \lor b \neq v \) — thus \( x \lor b = x \lor c \). This is impossible, since \( b \lor c = v \nleq x \). Thus there can be no such \( x \). \( \square \)

We next look at covers of the atoms of \( K \). If \( v \) is maximal in \( S \), then, since \( S \) then has no upper bound but \( v \), the only cover of any atom of \( K \) is \( v \). If \( v \) is not maximal in \( S \), then there is a covering sublattice \( K_1 = \{ u_1, a_1, b_1, c_1, v_1 \} \cong M_3 \) with, say, \( u_1 = a \) and \( c_1 = v \). Then \( a \) has exactly three covers—\( a_1, b_1 \), and \( v \). The element \( v \) is also a cover of \( b \) and \( c \). In addition, there is at most one cover not in \( S \):
Lemma 25. Let $K = \{ u, a, b, c, v \}$ and $K_1 = \{ u_1 = a, a_1, b_1, c_1 = v \}$ be covering $M_3$-s. Then there is at most one element $d_1 \in L$ distinct from $v$ that is a cover of $b$ or of $c$. Such a $d_1$ does not cover both $b$ and $c$. If there is such an element $d_1$, then there is a sublattice $K_2 = \{ u_2 = v, a_2 = v_1, b_2, c_2 = v_2 \}$, a covering $M_3$ with $d_1 < c_2$. (See Figure 14.) Furthermore, the only cover in $L$ of $d_1$ is $c_2$.

Proof. Without loss of generality, we may assume that $d_1$ covers $c$. Now, $d_1 \not\geq v_1$, for, by the dual of Lemma 24, the only lower covers of $v_1$ are $a_1$, $b_1$, and $v$. Thus $v_1$ is not maximal in $S$; we thereby get the covering sublattice $K_2 \cong M_3$. Set $a_2 = v_1$. Since $v$ and $d_1$ cover $c$, and $d_1 \not\geq v$, $v \lor d_1$ covers $v$, the zero of $K_2$. By Lemma 24, $v \lor d_1$ must be one of the atoms of $K_2$, and it cannot be $a_2 = v_1$. Set $c_2 = v \lor d_1$, and let $b_2$ be the third atom of $K_2$.

Now, $a_1$, $b_1$, $v$, and $d_1$ are all of height 2 above $u$. Thus, by width $\leq 4$, $c$ can have no other cover but $v$ and $d_1$, and $b$ can have no other cover but $v$; $d_1$ is the only other possible cover, but $b \lor c = v \neq d_1$.

Finally, $d_1$ can have no other cover than $c_2$. For if $x$ were a cover of $d_1$ distinct from $c_2$, then $\{ a_1, b_1, b_2, c_2, x \}$ would be a five-element antichain; for $a_1 \leq x$ implies that $x = a_1 \lor d_1$, which implies that $x$ is a lower cover of $v_2$, contradicting the dual of Lemma 24, and similarly for $b_1$. (See Figure 14.)

As in the Introduction, such a $d_1$ is the new element $d_{\nu, \varphi}$ where $\varphi(v) = \langle c, c_2 \rangle$. Since all intervals with end-points in $S$ are of finite length, any interval in $L$ with endpoints in $S$ is then contained in some $\text{Snake}(n, \varphi')$, determined by some $n$-snake.
$S'$ in $S$. Let us denote by $\overline{S}$ the union of all intervals in $L$ with end-points in $S$. Then $\overline{S}$ is one of $\text{Snake}(n, \varphi)$, $\text{Snake}(\uparrow, \varphi)$, $\text{Snake}(\downarrow, \varphi)$, $\text{Snake}(\rightarrow, \varphi)$ of the Introduction.

**Lemma 26.** Let $K$ be a covering $M_3$ in $S$ with zero $u$ and unit $v$. Let $C$ be a chain in $\overline{S}$. Then $C$ contains at most 4 elements that are $\geq u$ but not $> v$.

**Proof.** We need only refer to Figure 15, where the dashed circles are elements that may or may not be in $\overline{S}$, and, in contrast to earlier figures, dashed lines denote coverings that “are there” if the dashed circles are.

As a consequence we have:

**Corollary 27.** Let $x \in \overline{S}$ and let $C$ be an infinite chain in $\overline{S}$ of elements $\geq x$. Then the unit $v$ of any covering $M_3$ above $x$ is bounded above by infinitely many elements of $C$.

As shown above, there can be a single upper bound of $\overline{S}$ not in $\overline{S}$ and a single lower bound of $\overline{S}$ not in $\overline{S}$. We now show that any other element of $L$ is an element of $\overline{S}$.

**Lemma 28.** Let $x \in L$ be neither an upper bound nor a lower bound of $S$. Then $x \in \overline{S}$.

**Proof.** By duality, it suffices to show the following: any $x \in L$ that is not an upper bound of $S$ is bounded above by some element of $S$. Since $x$ is not an upper bound of $S$, it is not maximal in $L$. There is thus a $y \in L$ with $x < y$. There is then a prime interval $[x_1, y_1]$ with $x_1 \geq x$. It suffices to show that $x_1$ is bounded above by an element of $S$. By Lemma 22, $[x_1, y_1]$ is either perspective up to a lower edge of a covering $M_3$ in $S$ or perspective down to an upper edge of a covering $M_3$ in $S$. In the first eventuality $x$ is then $\leq$ an element of $S$. 

![Figure 15. Illustrating the proof of Lemma 26.](image-url)
We then need only consider the case where \([x_1, y_1]\) is perspective down to an upper edge \([s, t]\) of a covering \(M_3\) in \(S\). Assume, to the contrary, that \(x_1\) is not \(\leq\) any element of \(S\). Then, of course, \(s < x_1\).

Let \(s \leq x' < x_1\). Then there is a prime interval \([x_2, y_2]\) with \(x' \leq x_2 < y_2 \leq x_1\), and \([x_2, t \vee x_2]\) is a prime interval that is perspective down to \([s, t]\). Thus, by Lemma 4 and Lemma 8, \([x_2, y_2]\) is perspective up to a lower edge of a covering \(M_3\) in \(L\), and so \(y_2\) is \(\leq\) some element of \(S\). Consequently, any \(x'\) with \(s \leq x' < x_1\) is \(\leq\) some element of \(S\). Let \(C\) be a maximal chain in the set \(I = [s, x_1]\). If \(C\) contains a maximal element \(c\), then \([c, x_1]\) is a prime interval, and so is perspective up to a prime interval in \(S\), as above. But then \(x_1\) is bounded above by an element of \(S\), contrary to our assumption. Thus \(C\) is certainly infinite. Now, each element of \(C\) is bounded above by some element of \(S\) and bounded below by \(s \in S\). Thus \(C \subseteq S\).

Then, by Corollary 27, the unit of any covering \(M_3\) that is \(\geq s\) is bounded above by some element of \(C\), and so by \(x_1\). But then \(x_1\) is an upper bound of \(S\), and then so is \(y_1\), contradicting Lemma 23, that states that \(S\) has at most one upper bound.

This contradiction shows that \(x_1\), and so \(x\), must have an upper bound in \(S\), concluding the proof.

Consequently, \(L\) consists of \(S\) and at most one upper bound of \(S\) and at most one lower bound of \(S\), and thus is isomorphic to one of the lattices in (iii) – (vi) of Freese’s Structure Theorem in the Introduction, concluding the proof of that Theorem.

\section*{REFERENCES}

[8] G. Grätzer and D. Kelly, The variety generated by planar modular lattices, manuscript.