NOTES ON SECTIONALLY COMPLEMENTED LATTICES. III.
THE GENERAL PROBLEM

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Abstract. In a recent survey article, G. Grätzer and E. T. Schmidt raise the problem when is the ideal lattice of a sectionally complemented chopped lattice sectionally complemented. The only general result is a 1999 lemma of theirs, stating that if the finite chopped lattice is the union of two ideals that intersect in a two-element ideal \( U \), then the ideal lattice of \( M \) is sectionally complemented.

In this paper, we present examples showing that in many ways their result is optimal. A typical result is the following: For any finite sectionally complemented lattice \( U \) with more than two elements, there exists a finite sectionally complemented chopped lattice \( M \) that is (i) the union of two ideals intersecting in the ideal \( U \); (ii) the ideal lattice of \( M \) is not sectionally complemented.

1. Introduction

This paper continues the series of notes on sectionally complemented lattices, see G. Grätzer and H. Lakser [4] and [5]. While these earlier papers concentrated on the sectional complement construction in G. Grätzer and E. T. Schmidt [6] (the “1960 sectional complement”), now we investigate the one useful general tool (G. Grätzer and E. T. Schmidt [7]) to construct congruence preserving extensions of finite lattices to sectionally complemented lattices:

**Atom Lemma.** Let \( L_1 \) and \( L_2 \) be (disjoint) finite sectionally complemented lattices with zero elements \( 0_{L_1} \) and \( 0_{L_2} \), respectively. Let \( p_{L_1} \) be an atom of \( L_1 \) and let \( p_{L_2} \) be an atom of \( L_2 \).

Form the partial lattice \( M \) by identifying \( 0_{L_1} \) with \( 0_{L_2} \) and \( p_{L_1} \) with \( p_{L_2} \); after the identifications, \( L_1 \cap L_2 = \{0, p\} \), where \( p \) is an atom. Then \( M \) is a finite chopped lattice and \( \text{Id} M \) is a finite sectionally complemented lattice.

In a survey article on finite congruences, G. Grätzer and E. T. Schmidt [8] raise the following question (Problem 1):

Let \( M \) be a finite sectionally complemented chopped lattice. Under what conditions is \( \text{Id} M \) sectionally complemented?

In this paper we make various contributions to this problem, mostly negative ones.

First, we present an example to show that if in the Atom Lemma, we replace the condition \( L_1 \cap L_2 = \{0, p\} \) with \( |L_1 \cap L_2| = 4 \), then \( \text{Id} M \) is not necessarily sectionally complemented.

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complemented; see Theorem 1. In this example, \( L_1 = L_2 = L \) of Figure 1 and the chopped lattice \( M \) is shown in Figure 2.

We then generalize this example to \( L_1 \cap L_2 = U \), where \( U \) is any finite sectionally complemented lattice of more than two elements, see Theorem 3.

One may attempt to generalize the Atom Lemma by forming a chopped lattice \( M \) as a union of three ideals: \( L_1, L_2, \) and \( L_3 \), satisfying \( |L_1 \cap L_2| = |L_2 \cap L_3| = |L_3 \cap L_1| = 2 \). We show that, in general, \( \text{Id} M \) is not sectionally complemented with the example of Figure 3 (for \( i = 1, 2, 3 \)); see Theorem 4.

We conclude this note with a generalization of the Atom Lemma, see Theorem 5.

2. The Atom Lemma is best possible

2.1. Compatible pairs. In this section and the next, we shall consider chopped lattices \( M \) with two maximal elements \( m_1 \) and \( m_2 \). Let \( L_1 = [0, m_1] \) and \( L_2 = [0, m_2] \), and let \( u = m_1 \wedge m_2 \). An ideal \( I \) of \( M \) determines a pair \( \langle i_1, i_2 \rangle \), where \( i_j \) is the largest element of \( I \) in \( L_j \), for \( j = 1, 2 \). For such a pair,

\[
i_1 \wedge u = i_2 \wedge u.
\]

If this equation holds, we call \( \langle i_1, i_2 \rangle \) compatible.

It is easy to see that there is a one-to-one correspondence between ideals of \( M \) and compatible pairs, so instead of ideals, we shall work with compatible pairs. If \( \langle i_1, i_2 \rangle \) and \( \langle j_1, j_2 \rangle \) are compatible, then so is \( \langle i_1 \wedge j_1, i_2 \wedge j_2 \rangle \). Of course, \( \langle i_1 \lor j_1, i_2 \lor j_2 \rangle \) need not be compatible, but it is contained in the smallest compatible pair \( \langle i_1, i_2 \rangle \).

2.2. The example. We start with the lattice \( L \) of Figure 1.

**Lemma 1.** The lattices \( L \) is sectionally complemented.

**Proof.** By inspection. \( \square \)

We take two copies of \( L \), call them \( L_1 \) and \( L_2 \), and identify the four elements of the ideal \( \{0, p, q, u\} \) of \( L_1 \) with the corresponding elements on \( L_2 \), so that \( L_1 \cap L_2 = \{0, p, q, u\} \). We form the chopped lattice \( M = L_1 \cup L_2 \); see Figure 2, which also shows the lettering we use for the elements of \( L_1 \) and \( L_2 \).

**Lemma 2.** The ideal lattice of \( M \) is not sectionally complemented.

![Figure 1. The basic lattice L.](image)
Figure 2. The chopped lattice $M$.

Proof. Note that $p$ is meet-irreducible in $L_2$ and $q$ is meet-irreducible in $L_1$.

The unit element of the ideal lattice of $M$ is the compatible pair $\langle m_1, m_2 \rangle$. We show that the compatible pair $\langle a, b \rangle$ has no complement in the ideal lattice of $M$.

Assume, to the contrary, that the compatible pair $\langle s, t \rangle$ is a complement of $\langle a, b \rangle$. Since $\langle a, b \rangle \leq \langle a \lor u, m_2 \rangle$, a compatible pair, $\langle s, t \rangle \nleq \langle a \lor u, m_2 \rangle$, that is,

(1) $s \nleq a \lor u$.

Similarly, by considering $\langle m_1, b \lor u \rangle$, we conclude that

(2) $t \nleq b \lor u$.

Now, $\langle a, b \rangle \leq \langle p', q' \rangle$, a compatible pair. Then $\langle s, t \rangle \nleq \langle p', q' \rangle$, and so either $s \nleq p'$ or $t \nleq q'$. Without loss of generality, we may assume that

$s \nleq p'$.

It then follows by (1) that $s$ can be only $c$ or $m_1$. Then, since $s \land a = 0$,

$s = c$.

Thus $s \land u = p$, and so $t \land u = p$. But, in $L_2$, $p$ is meet-irreducible. Thus

$t = p \leq b \lor u$,

contradicting (2).

This proves:

**Theorem 1.** There is a finite chopped lattice $M$ with the following properties:

1. $M$ has exactly two maximal elements $m_1$ and $m_2$;
2. $M$ is sectionally complemented;
3. $[0, m_1 \land m_2]$ is the four-element Boolean lattice;
4. the ideal lattice $\text{Id} M$ of $M$ is not sectionally complemented.
3. Generalizing the example

In Section 2, we provided an example that in the Atom Lemma, the condition $L_1 \cap L_2 = \{0, p\}$, that is, $L_1 \cap L_2 = C_2$, the two-element chain, cannot be changed to $L_1 \cap L_2 = B_2$, the four-element Boolean lattice. In this section we generalize this by replacing the four-element Boolean lattice with any finite sectionally complemented lattice $U$ with more than two elements.

3.1. The $L\{U\}$ construction. Let $U$ be a finite sectionally complemented lattice with more than two elements, with zero 0 and unit $u$. Let $p$ and $q$ be distinct dual atoms of $U$. We form the disjoint union $L\{U\}$ of $L$ (of Figure 1) and $U$ and identify the four elements of the ideal $\{0, p, q, u\}$ of $L$ with the corresponding elements of $U$ so that $L \cap U = \{0, p, q, u\}$. Note that, in general, $\{0, p, q, u\}$ is not an ideal of $U$.

We define a binary relation $\leq$ on $L\{U\}$ as follows:

For $x, y \in L\{U\}$, let $x \leq y$ iff

(i) $x \leq_L y$, for $x, y \in L$;

(ii) $x \leq_U y$, for $x, y \in U$;

(iii) $x \leq_U z \leq_L y$, for $x \in U - \{0, p, q, u\}$, $y \in L - \{0, p, q, u\}$ and for some $z \in \{p, u\}$.

Lemma 3. $L\{U\}$ is a lattice and $L$ and $U$ are sublattices. For $x \in U - \{0, p, q, u\}$, $y \in L - \{0, p, q, u\}$,

$$x \lor y = \begin{cases} p \lor y, & \text{if } x \leq p; \\ u \lor y, & \text{if } x \not\leq p; \end{cases}$$

and

$$x \land y = \begin{cases} x, & \text{if } u \leq y; \\ x \land p, & \text{if } y = c; \\ 0, & \text{otherwise}. \end{cases}$$

Proof. Trivial. □

Lemma 4. The lattice $L\{U\}$ is sectionally complemented.

Proof. If $x \in [0, u]$, then $[0, x]$ is complemented because $U$ is sectionally complemented. In the remaining cases, we proceed by inspection. □

3.2. The general construction. Now we generalize Theorem 1 so that $[0, m_1 \land m_2] = U$, where $U$ is any finite sectionally complemented lattice. In fact, we shall prove more.

Let $U$ be a finite sectionally complemented lattice with more than two elements, with zero 0 and unit $u$. Let $p$ and $q$ be distinct dual atoms of $U$.

Let $L\{U\}$ be the lattice constructed in Section 3.1, with unit $m_1$.

Let $L_2$ be any sectionally complemented lattice with unit $m_2$ containing the ideal $U$ subject to the conditions: $u < m_2$ and $p$ is meet-irreducible in $L_2$.

Let $M$ be the chopped lattice, $M = L\{U\} \cup L_2$ and $L\{U\} \cap L_2 = U$.

Theorem 2. The ideal lattice, $\text{Id } M$, of $M$ is not complemented.

Proof. Since $a \land u = 0$, the pair $\langle a, 0 \rangle$ is compatible. We show that $\langle a, 0 \rangle$ has no complement in $\text{Id } M$.

Assume to the contrary that $\langle x, y \rangle$ is a complement of $\langle a, 0 \rangle$. Then

$$x \land a = 0.$$
The pair \( \langle m_1, u \rangle \) is compatible and \( \langle a, 0 \rangle \leq \langle m_1, u \rangle \). Thus \( \langle x, y \rangle \nleq \langle m_1, u \rangle \), that is,

\[ y \nleq u. \]

The pair \( \langle a, 0 \rangle \leq \langle a \lor u, m_2 \rangle \), which is compatible. So \( \langle x, y \rangle \nleq \langle a \lor u, m_2 \rangle \), that is, \( x \nleq a \lor u \).

By (3), either

\[ x = c \]

or

\[ x = c \land p'. \]

If (5) holds, then \( x \land u = p \) and since \( \langle x, y \rangle \) is compatible, we conclude that \( y \land u = p \). However, by assumption, \( p \) is meet-irreducible in \( L_2 \). Thus \( y = p \), contradicting (4). Thus (5) cannot hold.

Therefore, (6) holds. It follows that \( u \land x = 0 \) and since \( \langle x, y \rangle \) is compatible, we conclude that \( u \land y = 0 \). So \( \langle p', y \rangle \) is compatible. Since \( \langle a, 0 \rangle \leq \langle p', y \rangle \), and \( \langle x, y \rangle = \langle c \land p', y \rangle \leq \langle p', y \rangle \) by (6), therefore,

\[ \langle a, 0 \rangle \lor \langle x, y \rangle \leq \langle p', y \rangle < \langle m_1, m_2 \rangle, \]

showing that \( \langle x, y \rangle \) is not a complement of \( \langle a, 0 \rangle \). □

**Theorem 3.** Let \( U \) be a finite sectionally complemented lattice with more than two elements. Then there is a finite chopped lattice \( M \) with the following properties:

1. \( M \) has exactly two maximal elements \( m_1 \) and \( m_2 \);
2. \( M \) is sectionally complemented;
3. \( [0, m_1 \land m_2] \) is isomorphic to \( U \);
4. the ideal lattice \( \text{Id} M \) of \( M \) is not sectionally complemented.

**Proof.** This follows from Theorem 2; we only have to choose an \( L_2 \). The smallest example is \( L_2 = U \cup \{ u', m_2 \} \), where \( u' \) is a complement of \( u \). □

4. Cycles

In the Atom Lemma, we could take three finite sectionally complemented lattices \( L_1 \), \( L_2 \), and \( L_3 \) and create a chopped lattice \( M \) that is a union of these and assume that \( L_1 \cap L_2 \) and \( L_2 \cap L_3 \) are two-element ideals, while \( L_1 \cap L_3 = \{ 0 \} \). Then \( M \) is a finite sectionally complemented chopped lattice and \( \text{Id} M \) is a finite sectionally complemented lattice. The proof follows trivially by applying the Atom Lemma twice.

A novel situation arises, however, if we allow a “cycle”, that is, \( L_1 \cap L_2, L_2 \cap L_3, \) and \( L_3 \cap L_1 \) are all two-element ideals. Indeed, there is nothing special about three—we can take any number, more than two, of lattices.

**Theorem 4.** Let \( n \) be an integer \( \geq 3 \). There is a finite chopped lattice \( M \) with the following properties:

1. \( M \) has exactly \( n \) maximal elements \( m_1, m_2, \ldots, m_n \);
2. \( M \) is sectionally complemented;
3. if \( i - j \equiv 1 \pmod{n} \), then \( m_i \land m_j \) is an atom of \( M \), for distinct \( i, j \leq n \); otherwise, \( m_i \land m_j = 0 \);
4. the ideal lattice \( \text{Id} M \) of \( M \) is not sectionally complemented.
Proof. Let us define the lattice $L_i$, for $i = 1, 2, \ldots, n$, as shown in Figure 3. All the subscripts in the figure are taken modulo $n$; for example, if $i = n$, then $p_{i,i+1} = p_{n,1}$.

Let $M$ be the chopped lattice with $n$ maximal elements $m_1, m_2, \ldots, m_n$, so that in $M$ the following three conditions hold, for each $i \in \{1, \ldots, n\}$:

1. $[0, m_i] = L_i$;
2. $m_i \land m_{i+1} = p_{i,i+1} = p_{i+1,i}$;
3. if $i, j$ are distinct and $|i - j| \not\equiv 1 (\text{mod } n)$, then $m_i \land m_j = 0$.

An ideal $I$ of $M$ determines an $n$-tuple $\langle i_1, \ldots, i_n \rangle$, where $i_j$ is the largest element of $I$ in $M_j$, for $j \in \{1, \ldots, n\}$. For such an $n$-tuple,

$$i_j \land p_{j,j+1} = i_{j+1} \land p_{j+1,j}, \text{ for all } j \in \{1, \ldots, n\},$$

(Recall, in $M$, $p_{j,j+1} = p_{j+1,j}$.) We call such an $n$-tuple $\langle i_1, \ldots, i_n \rangle$ compatible.

It was verified in [3] that there is a one-to-one correspondence between ideals and compatible $n$-tuples; so instead of ideals, we shall work with compatible $n$-tuples.

Observe that $M$ is sectionally complemented.

We shall verify the theorem by proving that the compatible $n$-tuple (ideal) $\langle a_1, \ldots, a_n \rangle$ has no complement in the ideal lattice of $M$.

Let us assume, to the contrary, that the compatible $n$-tuple $\langle x_1, \ldots, x_n \rangle$ is a complement of $\langle a_1, \ldots, a_n \rangle$. So

$$\langle a_1, \ldots, a_n \rangle \land \langle x_1, \ldots, x_n \rangle = \langle a_1 \land x_1, \ldots, a_n \land x_n \rangle = \langle 0, \ldots, 0 \rangle.$$ 

From Figure 3, we conclude that

$$x_i \in \{0, p_{i,i-1}, p_{i,i+1}, e_i, r_i, s_i\}, \text{ for each } i \in \{1, \ldots, n\}.\quad (7)$$

Let us now assume that $x_i \leq e_i$, for some $i \in \{1, \ldots, n\}$. Then

$$\langle x_1, \ldots, x_n \rangle \leq \langle m_1, \ldots, m_{i-1}, e_i, m_{i+1}, \ldots, m_n \rangle.$$ 

Since also

$$\langle a_1, \ldots, a_n \rangle \leq \langle m_1, \ldots, m_{i-1}, e_i, m_{i+1}, \ldots, m_n \rangle,$$

and $\langle m_1, \ldots, m_{i-1}, e_i, m_{i+1}, \ldots, m_n \rangle$ is compatible, this contradicts that $\langle x_1, \ldots, x_n \rangle$ is a complement of $\langle a_1, \ldots, a_n \rangle$.

So $x_i \not\leq e_i$, for each $i \in \{1, \ldots, n\}$. It follows from (7) that $x_i \in \{r_i, s_i\}$, for each $i \in \{1, \ldots, n\}$.

![Figure 3. The lattice $L_i$.](image-url)
Let us assume that \( x_i = s_i \), for some \( i \in \{1, \ldots, n\} \). Then \( x_i \geq p_{i,i+1} \). Since \( \langle x_1, \ldots, x_n \rangle \) is compatible, \( x_{i+1} \geq p_{i+1,i} \). Thus, by (7), either \( x_{i+1} = p_{i+1,i} \) or \( x_{i+1} = c_{i+1} \), both contradicting that \( x_{i+1} \not\leq e_{i+1} \), proved above.

We are then left with a single case: \( \langle x_1, \ldots, x_n \rangle = \langle r_1, \ldots, r_n \rangle \). But \( \langle a_1, \ldots, a_n \rangle \vee \langle r_1, \ldots, r_n \rangle = \langle u_1, \ldots, u_n \rangle \), a compatible \( n \)-tuple.

We conclude that \( \langle a_1, \ldots, a_n \rangle \) has no complement in the ideal lattice of \( M \). □

5. Generalizing the Atom Lemma

Let \( M \) be a finite chopped lattice with \( n \) maximal elements \( m_1, m_2, \ldots, m_n \), satisfying the condition that for \( i < j \leq n \), the element \( m_i \land m_j \) is either 0 or an atom of \( M \). We associate with \( M \) a graph \( G \) on the set \( \{1, 2, \ldots, n\} \): the vertices \( i \) and \( j \) are connected iff \( m_i \land m_j \) is an atom. In the Atom Lemma, \( G \) is defined on the set \( \{1, 2\} \), and 1 and 2 are connected by an edge.

The following result generalizes the Atom Lemma:

**Theorem 5.** Let \( G \) be a finite graph. All finite sectionally complemented chopped lattices \( M \) associated with \( G \) have the property that the ideal lattices \( \text{Id} M \) are sectionally complemented iff \( G \) is a tree.

**Proof.** Assume, first, that \( G \) is not a tree. Then \( G \) contains a minimal cycle \( v_1, \ldots, v_k \) with \( 2 < k \)—there is an edge in \( G \) joining \( v_i \) and \( v_j \) iff \( i \equiv j \pmod{n} \). We construct the sectionally complemented chopped lattice \( \langle 0, m_1 \rangle \cup \langle 0, m_2 \rangle \cup \cdots \cup \langle 0, m_k \rangle \) as in Theorem 4, and then enlarge it to a sectionally complemented chopped lattice \( M \) as follows:

For each edge \( e \) of \( G \), we choose an atom \( p_e \) of \( M \); if \( e \) is the edge joining \( v_i \) and \( v_{i+1} \) (the subscripts taken modulo \( k \)), then we set \( p_e = p_{i,i+1} \), and if \( e \) is any other edge, then we adjoin a new atom \( p_e \). For each vertex \( v \not\in \{v_1, \ldots, v_k\} \), we adjoin a new element \( m_v > 0 \) to \( M \) and set \( p_e < m_v \), for each edge \( e \) incident with \( v \). Thus, if \( v \) is an isolated vertex, then \( m_v \) is an atom. Finally, in order that \( M \) be sectionally complemented, if \( v \) has degree 1, that is, if \( v \) is incident with exactly one edge of \( G \), then we add a new atom \( a_v \) to \( M \) with \( a_v < m_v \). The resulting chopped lattice \( M \) is thereby sectionally complemented and associated with \( G \). Then \( N = [0, m_1] \cup [0, m_2] \cup \cdots \cup [0, m_k] \) is an ideal of \( M \) and the interval \( [0, N] \) of \( \text{Id} M \) is not complemented by Theorem 4—so \( \text{Id} M \) is not sectionally complemented.

Conversely, let us assume that \( G \) is a tree, and let \( M \) be a sectionally complemented chopped lattice associated with \( G \). We proceed by induction on \( |G| = n \). The base of the induction is provided by the Atom Lemma.

If there are no edges in \( G \), then \( \text{Id} M \) is clearly sectionally complemented. If there are edges in \( G \), then choose the vertices \( v_1 \neq v_2 \) connected by an edge, so that this edge is the only edge from \( v_1 \)—we can do this because \( G \) is a tree. Define \( M_1 = [0, m_{v_1}] \) and \( M_2 = \bigcup([0, m_v] \mid v \in G, v \neq v_1) \). Then \( M_1 \) is sectionally complemented and \( \text{Id} M_2 \) is sectionally complemented by the induction hypothesis.

Applying the Atom Lemma to \( M_1 \) and \( \text{Id} M_2 \), we obtain a sectionally complemented chopped lattice, whose ideal lattice is sectionally complemented by the Atom Lemma. This ideal lattice is isomorphic to \( \text{Id} M \). □

**References**


