NOTES ON TOLERANCE RELATIONS OF LATTICES:  
A CONJECTURE OF R.N. McKENZIE*

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1. Introduction

A tolerance relation $T$ on a lattice $L$ is defined as a reflexive and symmetric binary relation having the substitution property. A maximal $T$-connected subset of $L$ is a $T$-block. The quotient lattice $L/T$ consists of the $T$-blocks with the natural ordering.

If $V$ and $W$ are lattice varieties, their product $V \circ W$ consists of all lattices $L$ for which there is a congruence relation $\Theta$ satisfying: (i) all $\Theta$-classes of $L$ are in $V$; (ii) $L/\Theta$ is in $W$. In general, $V \circ W$ is not a variety; however, $H(V \circ W)$ (the class of all homomorphic images of members of $V \circ W$) always is.

If $L$ is in $V \circ W$ (established by $\Theta$), then $L/\Phi$ is a typical member of $H(V \circ W)$. On $L/\Phi$, $\Theta/\Phi$ is a tolerance relation. The following theorem was conjectured by R.N. McKenzie: a lattice $K$ belongs to the variety generated by $V \circ W$ iff there is a tolerance relation $T$ on $K$ satisfying: (i) all $T$-classes of $L$ are in $V$; (ii) $L/T$ is in $W$.

In this paper we disprove this conjecture:

**Theorem.** The lattice $F$ of Fig. 1 is in $H(M_3 \circ D)$. However, there is no $A$ in $M_3 \circ D$ with congruence $\Theta$ establishing this such that $F$ can be represented as $A/\Theta$ and $T=\Theta/\Phi$ satisfies (i) all $T$-classes of $A$ are in $M_3$; (ii) $A/T$ is in $D$.

In this theorem, $M_3$ is the variety generated by the modular lattice $M_3$ and $D$ is the variety of all distributive lattices. The Theorem holds for some varieties other than $M_3$, see Section 5.

In Section 2 we introduce a new lattice construction, called *hinged-product* which

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we shall utilize to construct an interesting lattice. Since we hope that this construction will find other applications as well, we develop it in some detail.

In Section 3 we introduce the lattice $F$ of Fig. 1, and using a hinged-product (see Figs 2 and 3), we show that $F$ is in $M_3 \odot D$.

In Section 4 we investigate the tolerance relations of $F$; they form a lattice shown in Fig. 6.

Finally, in Section 5, we prove the Theorem.

For the basic concepts and unexplained notations, the reader is referred to [1].

2. Hinged-products

We start with a definition:

Definition 1. We are given a family, $L_i$, $i \in I$, of lattices; in each lattice $L_i$, we are given three elements, the hinge: $l_i \leq m_i \leq u_i$. The hinged-product $H = H(L_i, l_i, m_i, u_i)$ consists of the following subsets of the direct product of the $L_i$, $i \in I$:

(i) the ideal $l(H)$, the direct product of $(l_i)$, $i \in I$;
(ii) the dual ideal $u(H)$, the direct product of $(u_i)$, $i \in I$;
(iii) for every $i \in I$, the $i$th frame, $f_i(H)$, consisting of all elements whose $j$th coordinate is $m_j$ for all $j \neq i$.

$H$ is partially ordered componentwise.

Observe that these sets may not be disjoint: for instance, if $m_i = l_i$, then $l(H) \cap f_i(H)$ is non-empty; if $m_i = u_i$, then $u(H) \cap f_i(H)$ is non-empty. The element $\mu$ whose $i$th coordinate is $u_i$ for all $i \in I$ belongs to all $f_j(H)$, $j \in I$. 
Lemma 2. \( H \) is a lattice.

\textbf{Proof.} Let \( \alpha, \beta \in H \). Then the componentwise join, \( \alpha \lor \beta \), is always in \( H \) (hence it is also the join in \( H \)) with one possible exception: \( \alpha \in f_i(H) \) and \( \beta \notin f_j(H), i \neq j \). Let \( \alpha = \langle a_k \rangle, \beta = \langle b_k \rangle \); now if \( a_i \lor m_i \geq m_i \) and \( b_j \lor m_j \geq m_j \), then \( \alpha \lor \beta \) has no upper bound in \( l(H) \), in any \( f_k(H) \), and it has a least upper bound, namely, \( \alpha \lor \beta \lor \mu \), in \( u(H) \). Hence, \( \alpha \) and \( \beta \) have a least upper bound in \( H \), namely, \( \alpha \lor \beta \lor \mu \). We can argue the meets dually. This completes the proof of Lemma 2. \( \square \)

We shall continue to denote componentwise join and meet with \( \lor \) and \( \land \); the join and meet in \( H \) will be denoted by \( \lor \) and \( \land \), respectively.

It is not very easy to visualize \( H \). The \( i \)th frame, \( f_i(H) \) is isomorphic to \( L_i \). The \( f_i(H) \) are glued together at the hinges: \( l_i, m_i, u_i \). The glued frames are completed into a lattice by \( l(H) \) and \( u(H) \). The example of Section 3 may help illuminate this point.

If we have homomorphisms \( \phi_i : L_i \to L'_i \), then under certain conditions these homomorphisms have a joint extension from the hinged-product \( H \) to the hinged-product \( H' \):

\textbf{Lemma 3.} Let \( H = H(L_i, l_i, m_i, u_i) \) and \( H' = H(L'_i, l'_i, m'_i, u'_i) \) be hinged-products with the same index set \( I \). Let \( \phi_i : L_i \to L'_i \) be homomorphisms for \( i \in I \). Let us assume that \( \phi_i(l_i) = l'_i, \phi_i(m_i) = m'_i, \) and \( \phi_i(u_i) = u'_i \). If (i) or (ii) below holds, then the restriction \( \phi \) of the product of the homomorphisms \( \phi_i, i \in I, \) to \( H \) is a homomorphism of \( H \) into \( H' \).

(i) For all \( i \in I, x_i \lor m_i > m_i \) implies that \( \phi(x_i) \lor m'_i > m'_i \); and the dual condition for \( \land \).

(ii) For all \( i \in I, \phi(l_i) = \phi(u_i) \).

\textbf{Proof.} Under the first condition, whenever \( \alpha \lor \beta \neq \alpha \lor \beta \), then \( \phi(\alpha) \lor \phi(\beta) \neq \phi(\alpha) \lor \phi(\beta) \); therefore, \( \alpha \lor \beta = \alpha \lor \beta \lor \mu \) and \( \phi(\alpha) \lor \phi(\beta) = \phi(\alpha) \lor \phi(\beta) \lor \phi(\mu) \). It now follows that \( \phi(\alpha) \lor \phi(\beta) = \phi(\alpha \lor \beta) \). The argument for the meet is dual.
Under the second condition, \( \phi(m_i) = \phi(u_i) \), so \( \phi(\alpha \lor \beta) = \phi(\alpha \lor \beta) \) is obvious.

Note that a necessary and sufficient condition for \( \phi \) to be a homomorphism can be easily formulated. We only need the sufficient conditions of Lemma 3 in this paper.

3. \( F \) is in \( \text{H}(M_3 \circ D) \)

We start with the lattice of Fig. 2. We take three copies of \( L, L_1, L_2, \) and \( L_3 \). The elements will be denoted accordingly: \( u_1, u_2, \) and so on. Let \( A \) denote the hinged-product (power) of \( L_1, L_2, \) and \( L_3 \). The diagram of \( A \) is given in Fig. 3; \( l(H) = \{0\}, u(H) = \{\mu\} \).

Now consider the congruence relation \( \Theta(0, \mu) \) of \( L \), where \( 0 = \langle l_1, l_2, l_3 \rangle \) and \( \mu = \langle m_1, m_2, m_3 \rangle = \langle u_1, u_2, u_3 \rangle \). The natural homomorphism of \( L \) onto \( L/\Theta(0, \mu) \) obviously satisfies (ii) of Lemma 3, hence \( A \) has a natural homomorphism onto the appropriate hinged-product of three copies of \( L/\Theta(0, \mu) \); this new lattice is isomorphic to \( (C_2)^3 \).

There are eight \( \Theta \)-classes: four are isomorphic to \( M_3 \), three to \( (C_2)^2 \), and one to \( (C_2)^3 \). Since the congruence classes of \( \Theta \) are either distributive or isomorphic to \( M_3 \), and \( A/\Theta \) is distributive, we conclude that \( A \) belongs to \( M_3 \circ D \).
Next consider the congruence $\Theta(\mu, k)$ on $L$, where $k = \langle k_1, k_2, k_3 \rangle$. $L/\Theta(\mu, k)$ is the lattice of Fig. 4. The hinged cube of that lattice is isomorphic to the lattice $F$ of Fig. 1. Hence $F \in H(M_3 \circ D)$. This proves the first sentence of the Theorem.

4. The tolerance relations of $F$

Fig. 5 represents $F$ with the tolerance relation $T = \Theta/\Theta(0, \mu)$. $T$ is a natural tolerance on $F$; unfortunately, $F/T$ is isomorphic to $M_3$, and it is not distributive. This makes the Theorem plausible. If there are $A, \Theta, \Phi$ such that $A \in M_3 \circ D$ is established by $\Theta$, and $T = \Theta/\Phi$ satisfies that (i) all $T$-classes of $A$ are in $M_3$; (ii) $A/T$ is in $D$, then $A$ must be the lattice of Fig. 3 (or a fatter version with larger distributive classes), and $\Theta$ and $\Phi$ must be as in Section 3.

In Section 5 we shall prove this. As a first step, we have to describe all the tolerance relations on $F$. 

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Notes on tolerance relations of lattices

Fig. 4.

Fig. 5.
Fig. 6.

**Lemma 4.** The lattice $F$ has nine tolerance relations; they form a lattice as shown in Fig. 6. The tolerance $T_0$ is depicted in Fig. 5, and the tolerance $T_1$ in Fig. 7.

**Proof.** Since $F$ is a finite, simple, modular lattice, any two prime intervals of $F$ are projective. Therefore, $F$ has a unique minimal proper tolerance relation, $T_0$, generated by any prime interval. Moreover, if any two distinct elements of a sublattice isomorphic to $M_3$ are collapsed by a tolerance relation, then the whole sublattice is collapsed. It follows immediately that $T_0$ is as described in Fig. 5.

Next consider the tolerance relation $T_1$ of Fig. 7. Symmetrically, we can define $T_2$, and $T_3$. We shall prove that any proper tolerance relation is either a $T_i$ or of the form $T_i \lor T_j, i \neq j$.

Fig. 7.
So let $T$ be a tolerance relation, $T > T_0$. Then there are $x, y \in F$, $x < y$, such that $x \equiv y (T)$ but not modulo $T_0$.

**Claim.** $0 \equiv d_i (T)$ for some $i$.

**Proof.** Up to symmetry, there are three cases to consider:

- **Case 1.** $a_1 \leq x < d_1$ and $d_1 \lor d_2 \leq y$. Then
  
  \[ b_2 = b_2 \land (d_1 \lor d_2) = b_2 \land x = 0 \quad (T). \]

  Similarly, $c_3 \equiv 0 (T)$, so $d_2 \equiv 0 (T)$, as claimed.

- **Case 2.** $a_1 = x$ and $d_3 \leq y$. Then
  
  \[ 0 = 0 \lor 0 = (x \land b_3) \lor (x \land c_3) = (y \land b_3) \lor (y \land c_3) = b_3 \lor c_3 = d_3 \quad (T) \]

  as claimed. This completes the proof of the Claim.

- **Case 3.** $x = 0$ and $b_3 \leq y$. Then
  
  \[ 0 = 0 \lor 0 = x \lor x = (x \land b_3) \lor (((x \land a_3) \lor a_2) \land a_1) \]

  \[ = (y \land b_3) \lor (((y \land a_3) \lor a_2) \land a_1) = b_3 \lor a_1 = d_3 \quad (T) \]

  as claimed. This completes the proof of Lemma 4. The relation $d_1 \equiv 0$ obviously generates the tolerance $T_1$. Similarly we get $T_2$, and $T_3$.

Finally, let $T$ be a tolerance relation satisfying $T > T_1$. Then we must have $x \equiv y (T)$, $x < y$, such that $[x, y]$ properly contains a $T_1$-block. We distinguish four cases according to which $T_1$-block $[x, y]$ contains.

- **Case 1.** The block $[u, 1]$. $x < u$ and $1 \leq y$ imply that, say, $x \leq a_2$ and $y = 1$. Thus $a_2 \equiv 1 (T)$. Hence,

  \[ 0 = 0 \lor 0 = (c_1 \land a_2) \lor (c_3 \land a_2) = (c_1 \land 1) \lor (c_3 \land 1) = d_1 \lor d_3 \quad (T). \]

- **Case 2.** The block $[0, d_i]$. $x \leq 0$ and $d_i \leq y$ imply that $x = 0$ and, say, $d_1 \lor d_3 \leq y$. Thus $0 \equiv d_1 \lor d_3 (T)$. Hence,

  \[ a_2 = b_2 \land c_2 = (0 \lor b_2) \land (0 \lor c_2) = ((d_1 \lor d_3) \lor b_2) \land ((d_1 \lor d_3) \lor c_2) = 1 \quad (T). \]

- **Case 3.** The block $[a_2, d_3 \lor d_1]$. $x \leq a_2$ and $d_1 \lor d_3 \leq y$. Hence either $x = 0$ and $d_1 \lor d_3 \leq y$, in which case we proceed as in Case 2, or $x \leq a_2$ and $y = 1$, and we proceed as in Case 1.

- **Case 4.** The block $[a_3, d_1 \lor d_2]$. We proceed, by symmetry, as in Case 3.

Thus in all four cases we have $0 \equiv d_1 \lor d_3 (T)$ and $a_2 \equiv 1 (T)$ or $0 \equiv d_1 \lor d_2 (T)$ and $a_3 \equiv 1 (T)$. These, obviously, describe the tolerances $T_1 \lor T_2$ and $T_1 \lor T_2$, respectively.

Since $T_i \lor T_j$, $i \neq j$ are maximal tolerances, the proof of Lemma 4 is complete.
5. The proof of the Theorem

Let us assume that there is an $A$ in $M_3 \circ D$ with congruence $\Theta$ establishing this such that $F$ can be represented as $A/\Theta$ and $T=\Theta/\Phi$ satisfies (i) all $T$-classes of $A$ in $M_3$; (ii) $A/T$ is in $D$.

It is easy to see that the lattice $A/\Theta \wedge \Phi$ and the congruences $A/\Theta \wedge \Phi$ and $\Theta/\Theta \wedge \Phi$ satisfy the same conditions, and the new congruences are disjoint. In other words, we can assume that $\Theta \wedge \Phi = \omega$.

An element $x$ of $F$ is represented as a $\Phi$ congruence class, $C(x)$. $A/T$ is distributive; since $A/T_0$ is isomorphic to $M_3$, $T > T_0$. Thus by the Claim in the proof of Lemma 4, we can assume that $0 \equiv d_3 (T)$.

Hence there are elements $0' \in C(0)$ and $d'_3 \in C(d_3)$ satisfying $0' \equiv d'_3 (\Theta)$. We can obviously assume that $0' < d'_3$. Substituting an arbitrary $a'_3 \in C(a_3)$ by $(0' \lor a'_3) \land d'_3$, we obtain $a'_3 \in C(a_3)$ satisfying $0' < a'_3 < d'_3$. Similarly, we can choose $u' \in C(u)$, $b'_3 \in C(b_3)$, and $c'_3 \in C(c_3)$ satisfying $u', b'_3, c'_3 \in [a'_3, d'_3]$ and elements $a'_1 \in C(a_1)$, $a'_2 \in C(a_2)$ in $[0', u']$.

Now it is easy to see that the elements $0'$, $a'_1$, $a'_2$, $a'_3$, $u'$, $b'_3$, $c'_3$, and $d'_3$ form a sublattice of $A$ isomorphic to the interval $[0, d_3]$ of $F$. Indeed, the map $\phi : x \rightarrow x'$ is obviously one-to-one. Since $0' \equiv d'_3 (\Theta)$, all these elements belong to the same $\Theta$ class. We have to show that the $\lor$ and $\land$ work properly. As an example, let us show that $a'_1 \lor a'_2 = u'$. Indeed, $a'_1 \lor a'_2 = u' (\Theta)$ since all these elements are in the same $\Theta$ class. On the other hand, $a'_1 \lor a'_2 = u' (\Phi)$ since both $a'_1 \lor a'_2$ and $u'$ map onto $u$.

Therefore, $a'_1 \lor a'_2 = u' (\Theta \land \Phi)$. Since $\Theta \land \Phi = \omega$, we conclude that $a'_1 \lor a'_2 = u' (\omega)$, that is, $a'_1 \lor a'_2 = u'$, as claimed.

Since every $\Theta$ class is in $M_3$, we get that the interval $[0, d_3]$ of $F$ is in $M_3$, an obvious contradiction which proves the Theorem.

It is obvious from the proof, that the Theorem holds for any lattice variety $V$ in place of $M_3$ that does not contain the interval $[0, d_3]$ of $F$. The most general such variety is $M_\omega$, the lattice variety generated by $M_\omega$, the modular lattice of length two with countably infinite atoms.

References