Two Mal'cev-Type Theorems in Universal Algebra*

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ABSTRACT

Patterned after theorems of Mal'cev and Jónsson, certain types of conditions for equational classes of algebras are named "Mal'cev type." Regularity and weak regularity of equational classes of algebras are proved to be of "Mal'cev type."

1. Introduction

The classic theorem of Mal'cev [4] states that the congruences of any algebra of an equational class $K$ permute if and only if there exists a ternary polynomial symbol $p$ satisfying the identities

$$x_0 = p(x_0, x_1, x_1), \quad x_1 = p(x_0, x_0, x_1).$$

A similar result was proved Pixley [5] for permutability and distributivity. Distributivity alone was handled by Jónsson [3], and modularity by Day [1]. The following definition is patterned after the condition of Jónsson [3]:

Definition 1. A Mal'cev-type condition of equational classes is of the form "$(P_n)$ there exists a natural number $n$, and polynomial symbols $p_0, \ldots, p_{n-1}$ satisfying a set of identities $\Sigma_n$", where $(P_{n+1})$ is weaker than $(P_n)$ and the form of $\Sigma_n$ is independent of the type of algebras considered.

A. Tarski suggested that the following two properties of equational classes of algebras be considered:

Definition 2. An equational class $K$ of algebras is regular if every algebra $\mathcal{A} = \langle A; F \rangle \in K$ is regular in the following sense: If $\Theta$ and $\Phi$ are congruence relations of $\mathcal{A}$, and $\Theta$, $\Phi$ have a congruence class in common, then $\Theta = \Phi$.

Definition 3. An equational class $K$ of algebras is weakly regular if every algebra $\mathcal{A} \in K$ is weakly regular in the following sense: There

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exists an \( o \in A \) such that, if the congruences \( \Theta, \Phi \) have the same congruence class containing \( o \), then \( \Theta = \Phi \).

Sufficient conditions of Mal’cev type for regularity and weak regularity were found by A. Tarski (oral communication). Weak regularity with the further assumption that \( o \) be the value of a nullary polynomial was considered by Slomiński [6]. For an interesting result on weakly regular classes see Vaught [7].

The main result of this paper is the following:

**Theorem.** Regularity and weak regularity are equivalent to Mal’cev-type conditions.

The specific forms of the two theorems and their proofs yield a number of results on regularity and weak regularity. These will be given in the last section, along with some open problems.

The undefined concepts and notations are that of [2].

2. **Two Lemmas**

Lemmas 1 and 2 rephrase the definitions of regularity and weak regularity, respectively, in a form suitable for applications.

**Lemma 1.** The algebra \( \mathcal{A} \) is regular if and only if for \( a, b, c \in A \) there exist \( d_0, \ldots, d_{m-1} \in A \) (\( m \) may depend on \( a, b, c \)) such that

\[
\Theta(a, b) = \bigvee (\Theta(c, d_i) | 0 \leq i < m).
\]

**Proof:** Suppose that \( \mathcal{A} \) is regular, and take \( a, b, c \in A \). Set

\[
H = [c] \Theta(a, b).
\]

Then a trivial computation shows that

\[
[c] \Theta(a, b) = [c] \Theta(H);
\]

hence, by the definition of regularity, \( \Theta(a, b) = \Theta(H) \). Thus,

\[
\Theta(a, b) = \Theta(H) = \bigvee (\Theta(c, h) | h \in H),
\]

and, by the compactness of \( \Theta(a, b) \),

\[
\Theta(a, b) = \bigvee (\Theta(c, h) | h \in H_0),
\]

for some finite \( H_0 \subseteq H \), which is condition (1).

Conversely, assume (1), and take \( \Theta, \Phi \in C(\mathcal{A}) \), such that \( \Theta \) and \( \Phi \) have

\[
\Theta(a, b) = \left\{ x \mid x = a(\Theta) \right\}.
\]
one congruence class in common. Let \( c \) be an element of this class, and so \([c]o = [c]d\). Now let \( a \equiv b(\Theta)\). Then by (1), \( \Theta(a, b) = \lor \Theta(c, d)\). Thus \( c \equiv d_i(\Theta)\), \( i < m \), and so
\[
d_i \in [c]o = [c]d,
\]
that is, \( c \equiv d_i(\Theta)\), for all \( i < m \). This implies \( a \equiv b(\Theta)\). The same argument shows that \( a \equiv b(\Theta)\) implies \( a \equiv b(\Theta)\), hence \( \Theta = \Phi\), and \( \cal{O} \) is regular. The proof of Lemma 1 is complete.

**Lemma 2.** The algebra \( \cal{O} \) is weakly regular if there exists an \( o \in A\), such that for \( a, b \in A\) there exist \( d_i \in A\), \( i < m \), such that
\[
(2) \quad \Theta(a, b) = \lor (\Theta(o, d_i) \mid i < m).
\]
The proof is analogous to that of Lemma 1, and will therefore be omitted.

### 3. Regularity

In this section we will consider the following Mal'cev-type condition:
\((P_n)\). There exist ternary polynomial symbols \( p_i, 0 \leq i < n, 0 \leq j \leq n, \)
\( r_i, 0 \leq i < n, 0 \leq j < n, 1 \leq k < n, q_i, 0 \leq i < n, t_j, 0 \leq i < n, 1 \leq j < n, \)
and \( n\)-ary polynomial symbols \( r, 0 \leq i, j < n, \)
\( 0 \leq i < n \), such that \( p^0 = x_0, 0 \leq i < n, q_0 = x_0, q_n = x_1, \)
and such that the following set \( \Sigma_n \) of identities hold:
\[
\begin{align*}
(\Sigma_n, 1) & \quad r_i(x_0, r_i^i, \ldots, r_{n-1}^i) = p_i^i, \quad \text{for } 0 \leq i, j < n; \\
(\Sigma_n, 2) & \quad r_i(x_1, r_i^i, \ldots, r_{n-1}^i) = p_i^{i+1}, \quad \text{for } 0 \leq i, j < n; \\
(\Sigma_n, 3) & \quad t_i(x_2, t_i^i, \ldots, t_{n-1}^i) = q_i, \quad \text{for } 0 \leq i < n; \\
(\Sigma_n, 4) & \quad t_i(p_i, t_i^i, \ldots, t_{n-1}^i) = q_{i+1}, \quad \text{for } 0 \leq i < n.
\end{align*}
\]

An easy computation shows that \((P_n)\) satisfies the conditions of Definition 1. \("(P_n)\) implies \((P_{n+1})\" is essentially the same trick as the final step in the proof of Theorem 1, though a direct computation is equally easy.

\( \Sigma_n \) can be illustrated by Figure 1. The arrows indicate the way congruences are made to spread as a result of the identities \( \Sigma_n \), that is, 
\( x_0 \equiv x_1 \) implies \( p_i^i \equiv p_i^{i+1} \), and \( x_2 \equiv p_i^n \) implies \( q_i \equiv q_{i+1} \). \( r_i^j \) stands for \( r_i(x_0, r_i^j, \ldots) \) and \( t_i \) stands for \( t_i(x_0, t_i^j, \ldots) \).
THEOREM 1. An equational class $K$ of algebras is regular if and only if $(P_n)$ holds for some $n$.

PROOF: Let $(P_n)$ hold for $K$, and let $\mathcal{A} \in K$. The regularity of $\mathcal{A}$ will be shown by verifying (1) of Lemma 1. For $a, b, c \in A$ define

$$d^n = p_i(a, b, c), \quad 0 \leq i < n.$$ 

We claim that (1) holds for the $d_i$.

Substituting $a, b, c$ for $x_0, x_1, x_2$ in $(\Sigma_n, 1)$ and $(\Sigma_n, 2)$ we have

$$r_i(a, r_i^{i-1}(a, b, c), \ldots, r_i^{i-2}(a, b, c)) = p_i(a, b, c),$$

$$r_i(b, r_i^{i-1}(a, b, c), \ldots, r_i^{i-2}(a, b, c)) = p_i^{i+1}(a, b, c).$$

This implies that

$$p_i(a, b, c) = p_i^{i+1}(a, b, c)(\Theta(a, b)), \quad \text{for } 0 \leq i < n.$$ 

Using $p_i(a, b, c) = c$, $p_i(a, b, c) = d_i$, and the transitivity of $\Theta(a, b)$, we conclude

$$c = d_i(\Theta(a, b)),$$

and so

$$\Theta(a, b) \geq \bigvee (\Theta(c, d_i)) 0 \leq i < n.$$ 

Now we make the same substitution in $(\Sigma_n, 3)$ and $(\Sigma_n, 4)$:

$$t_i(c, t_i^{i}(a, b, c), \ldots, t_i^{i-2}(a, b, c)) = q_i(a, b, c),$$

$$t_i(d_i, t_i^{i}(a, b, c), \ldots, t_i^{i-2}(a, b, c)) = q_{i+1}(a, b, c),$$

for $0 \leq i < n$. 

\begin{figure}
\centering
\begin{tikzpicture}
\node (A) at (0,0) {$A$};
\node (B) at (2,0) {$B$};
\node (C) at (0,-2) {$C$};
\node (D) at (2,-2) {$D$};
\draw[->] (A) -- (B) node[midway, above] {$r_1$};
\draw[->] (C) -- (D) node[midway, above] {$r_1$};
\end{tikzpicture}
\caption{Figure 1}
\end{figure}
This implies that
\[ q_i(a, b, c) = q_{i+1}(a, b, c)(\Theta(c, d_i)), \quad 0 \leqslant i < n. \]

Using \( q_0(a, b, c) = a, q_n(a, b, c) = b \), we conclude that
\[ a = b \left( \bigvee (\Theta(c, d_i) \mid 0 \leqslant i < n) \right), \]
that is,
\[ \Theta(a, b) \leqslant \bigvee (\Theta(c, d_i) \mid 0 \leqslant i < n), \]
completing the proof of (1).

Conversely, let \( K \) be regular, and apply Lemma 1 to the algebra \( \mathcal{F} = \mathcal{F}_K(3) \), freely generated by \( \{ x_0, x_1, x_2 \} \) over \( K \). Set \( a = x_0, b = x_1, c = x_2 \); by Lemma 1 there exist \( d_i, 0 \leqslant i < m \), with
\[ \Theta(a, b) = \bigvee (\Theta(c, d_i) \mid 0 \leqslant i < m) \]
This is equivalent to the following two conditions:

\[ a = b \left( \bigvee (\Theta(c, d_i) \mid 0 \leqslant i < m) \right), \]
\[ c = d_i(\Theta(a, b)), 0 \leqslant i < m. \]

According to Theorem 10.4 of [2], (3) and (4) can be rewritten as follows:

\[ (3') \quad \text{There is a sequence } q_0 = a, ..., q_m = b \text{ of elements of } \mathcal{F}, \text{ and a sequence of unary algebraic functions } t_i \text{ such that } t_i(e) = q_i, t_i(d_i) = q_{i+1}, \text{ where } 0 \leqslant i < m. \]

\[ (4') \quad \text{There is a sequence of elements } p_0 = c, ..., p_{m+1} = d_i, \text{ and a sequence } r_i \text{ of unary algebraic functions such that } r_i(a) = p_i, r_i(b) = p_{i+1}. \]

Now the derivation of \((1, 1') - (1, 4')\) is straightforward. First, we write out the unary algebraic functions as polynomials with elements of \( \mathcal{F} \) substituted for all but the first variable. Then we replace all elements \( s \) of \( \mathcal{F} \) by a ternary polynomial symbol \( s \), which is mapped onto \( s \) under the natural homomorphism of \( \mathcal{F}^{\mathbb{N}(1')} \) onto \( \mathcal{F} \), and similarly we replace all polynomials by polynomial symbols. Finally, let \( n \) denote the maximum of all integers occurring in \((3')\) and \((4')\) as lengths of sequences or arities of polynomials, or occurring as arities of polynomials at the rewriting of algebraic functions. Now, observe that a \( k \)-ary polynomial symbol is also \( m \)-ary for \( k \leqslant m \); also, any sequence in \((3')\) and \((4')\) can be made
longer by repeating one of its terms. Finally, by allowing further repetitions, by renumbering we can obtain \( j_i = i \) in (3'). The resulting identities will take the form \((\Sigma_n', 1) - (\Sigma_n', 4)\). This concludes the proof of Theorem 1.

4. WEAK REGULARITY

In this section we will consider the Mal'cev-type condition \((Q_n)\) that we obtain from the condition of Section 3 by further postulating the existence of a unary polynomial symbol \( o \) in \( x_2 \) satisfying the identities \( \Omega_n \) we obtain from \( \Sigma_n \) by replacing all occurrences of \( x_2 \) by \( o \); that is, \( p^o = o \), and \( t_i(o, t_1', \ldots, t_{n-1}') = q_i \). Let \((\Omega_n', 3)\) stand for this identity, and let \((\Omega_n', i)\) stand for \((\Sigma_n', i), i = 1, 2, 4\).

**THEOREM 2.** An equational class \( K \) is weakly regular if and only if \( K \) satisfies \((Q_n)\) for some \( n \).

One would at first suspect that the proof of Theorem 2 is analogous to that of Theorem 1. However, if we apply Lemma 2 to \( \mathcal{F}_K(3) \) then the \( o \) we obtain may depend on \( x_0, x_1, \) and \( x_2 \) and then when we substitute \( a \) and \( b \) we will get an \( o \) depending on \( a \) and \( b \). The following sketch of the proof will give all the steps that are different from the steps in the proof of Theorem 1.

Let \( K \) satisfy \((Q_n)\), \( \mathcal{A} \in K \). Pick an arbitrary element \( g \in A \) and define \( o = o(g) \). Lemma 2 is then applied the same way Lemma 1 was applied in the previous proof.

Now let us assume that \( K \) is weakly regular, and consider \( \mathcal{F}_K(\omega) \). Since \( \mathcal{F}_K(\omega) \) is weakly regular, it has a special element \( a_1 \). By renumbering the free generators \( x_i, i < \omega \), we can assume that \( a_1 \) is in the subalgebra generated by \( x_2, \ldots, x_{n-1} \). Let the elements \( d_i, i < n \), be given as in Lemma 2.

Map \( \mathcal{F}_K(\omega) \) onto \( \mathcal{F}_K(3) \) by extending the map \( x_j \mapsto x_j, j \leq 2 \), to a homomorphism \( p \). Set \( o = o(p), d_i = d_i p \). It is easy to check that \( \Theta(x_0, x_2) = \cup (\Theta(o, d_i) \mid i < n) \). Now we can proceed with \( x_0, x_1, o, \) and \( d_i \) as in the proof of Theorem 1.

5. CONCLUSIONS

The obvious objection to Mal'cev-type theorems is that the conditions are so complicated that they are impractical. Nobody would suggest the use of the theorems of this note to show the regularity of groups,
rings, or Boolean algebras. However, the form of the Mal'cev-type conditions yields information which would be hard to obtain otherwise.

**Corollary 1.** An equational class $K$ is (weakly) regular if and only if every algebra of $K$ generated by three elements is (weakly) regular.

This is best possible: in the equational class $L$ of lattices every lattice generated by two elements is regular, but $L$ is not (weakly) regular.

**Corollary 2.** Let $\mathcal{A} = \langle A; F \rangle$ be a finite algebra of finite type (i.e., both $A$ and $F$ are finite). Then there is an effective way to decide whether the equational class $K$ generated by $\mathcal{A}$ is (weakly) regular.

**Proof:** We form $\mathcal{P}(\mathcal{A})$, which is finite, since $\mathcal{A}$ is finite. We then check whether the condition of Lemma 1 (Lemma 2) holds in $\mathcal{P}(\mathcal{A})$. If it does not, $K$ is not (weakly) regular. If it does, then we derive that $\mathcal{A}$ satisfies $P_n$ (respectively, $Q_n$) the same way as we proceeded in the proof of Theorem 1 (Theorem 2). Hence $K$ also satisfies $P_n$ (respectively, $Q_n$), and therefore $K$ is regular.

The same proof yields this result for finitely many, rather than one, algebras.

Let $\mathcal{A}$ be a weakly regular algebra, and let $0(\mathcal{A})$ denote the set of all $o \in A$ satisfying the requirement of Definition 4. By Definition 4, $|0(\mathcal{A})| \geq 1$.

**Corollary 3.** Let $K$ be a weakly regular equational class and let $\mathcal{A} \in K$. Then $0(\mathcal{A}) \cap B$ is non-void for any subalgebra $B$ of $\mathcal{A}$.

Proof is obvious from the proof of Theorem 2.

Call an equational class idempotent if $f(x, \ldots, x) = x$ holds for all operations.

**Corollary 4.** An idempotent weakly regular equational class is regular.

**Proof:** By Corollary 3 and by idempotency, $0(\mathcal{A}) = A$ for all $\mathcal{A} \in K$; $0(\mathcal{A}) = A$ is regularity.

Consider a type of algebras with operation $\wedge$, $\vee$ and $'$, all binary. Interpret these for congruences as join, meet, and relation-theoretic product. Let $p$ and $q$ be polynomials of this type.

**Problem.** Is the condition that the congruences of the algebras of an equational class $K$ satisfy $p = q$ equivalent to a Mal'cev-type condition? The special case, when $p$ and $q$ are formed by $\wedge$ and $'$ only, has been
settled, and the answer is always in the affirmative. This contains as a special case Mal’cev’s theorem, mentioned in the introduction, and also the following result.

Let us say that the congruences of $K$ are of type $n$, if for any $\Theta, \Phi \in \mathcal{C}(\mathcal{O})$, $\mathcal{O} \in K$,

$$\Theta \lor \Phi = \Theta \cdot \Phi \cdot \Theta \ldots \text{n-times}.$$ 

Then there is a Mal’cev-type condition equivalent to “the congruences of $K$ are of type $n$.” (Mal’cev’s theorem is the special case $n = 2$.)

Proving or disproving that a condition $P$ on equational classes is equivalent to a Mal’cev-type condition the following observations can be used.

If $P$ is equivalent to a Mal’cev-type condition, then

(i) if $K$ has $P$, and $K_1$ is equivalent to $K$, then $K_1$ has $P$;
(ii) if $K$ has $P$, and $K_1 \subseteq K$, then $K_1$ has $P$;
(iii) if $K$ has $P$, and $K$ is a reduct of $K_1$, then $K_1$ has $P$;
(iv) if $K_0$ and $K_1$ have $P$, so does $K_0 \times_f K_1$.

In (iv) $K_0 \times_f K_1$ is defined as follows: take $\mathcal{O} \in K_0$, $\mathcal{B} \in K_1$, and on $A \times B$ define an $n$-ary operation $\langle p, q \rangle$ componentwise for any $n \geq 0$, and for any pair of $n$-ary polynomials $p \in P_{\mathcal{O}}(\mathcal{O}), q \in P_{\mathcal{B}}(\mathcal{B})$; that is

$$\langle p, q \rangle (\langle a_0, b_0 \rangle, \ldots, \langle a_{n-1}, b_{n-1} \rangle) = \langle p(a_0, \ldots, a_{n-1}), q(b_0, \ldots, b_{n-1}) \rangle.$$ 

The resulting algebra is $\mathcal{O} \times_f \mathcal{B}$. Let $K_0 \times_f K_1$ be the equational class (of a suitable type) generated by all the $\mathcal{O} \times_f \mathcal{B}, \mathcal{O} \in K_0, \mathcal{B} \in K_1$.

Observations (i)-(iii) are obvious, and (iv) is also easily verified.

It is an open problem whether (i)-(iv) along with some further conditions of this type could characterize properties that are equivalent to Mal’cev-type conditions.

Finally, it should be emphasized that, even though in all the four Mal’cev-type theorems (Jónsson’s, Day’s, and the two theorems of the paper) it is intuitively clear that $(P_n)$ is not equivalent to $(P_{n+1})$, suitable examples to show this are not known.

Note Added in Proof. In Section 3, formulas (3) and (4) are rewritten using Theorem 10.4 of [2]; however the rewritten versions, (3') and (4') are stronger than what follows from Theorem 10.4 of [2]. In (3') we can only claim that $\{q_1, q_2\} = \{t(c), t(d)\}$, and

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5 In fact we always obtain a strong Mal’cev-type condition, i.e., one for which $\Sigma_1 = \Sigma_2 = \ldots$. 
in (4') that \( \{p_i, p^{\prime \prime i}\} = \{r_i(a), r_i(b)\} \). These statements cannot be translated readily into Mal'cev-type conditions.

Therefore, we should use the following version of Theorem 10.3 of [2]:

\[ x = \gamma(\Theta(a, b)) \text{ if, and only if, there exists an } n < \omega, \text{ a sequence } x = z_0, z_1, \ldots, z_n = y \text{ of elements, and a sequence } p_0, \ldots, p_{n-1} \text{ of unary algebraic functions such that } p_i(a) = z_i, \]

\[ p_i(b) = z_{i+1} \text{ for even } i, 0 < i < n, \text{ and } p_i(a) = z_i, p_i(b) = z_i \text{ for odd } i, 0 < i < n. \]

Theorem 10.4 of [2] should also be modified similarly.

Accordingly, \( \Sigma_n \) has to be revised:

\[
(\Sigma_n, 1) \ r^i(x, r_{i+1}, \ldots, r_{n-1}) = \begin{cases} p_i, & \text{for even } j \\ p_{i+1}, & \text{for odd } j \end{cases} \ 0 \leq i, j < n,
\]

\[
(\Sigma_n, 2) \ r^i(x, r_{i+1}, \ldots, r_{n-1}) = \begin{cases} p_{i+1}, & \text{for even } j \\ p_i, & \text{for odd } j \end{cases} \ 0 \leq i, j < n,
\]

\[
(\Sigma_n, 3) \ t(x, t_{i+1}, \ldots, t_{n-1}) = \begin{cases} q, & \text{for even } i \\ q_{i+1}, & \text{for odd } i \end{cases} \ 0 \leq i < n,
\]

\[
(\Sigma_n, 4) \ t(x, t_{i+1}, \ldots, t_{n-1}) = \begin{cases} q_{i+1}, & \text{for even } i \\ q_i, & \text{for odd } i \end{cases} \ 0 \leq i < n.
\]

Note that these changes do not affect the Theorem stated in the Introduction, the lemmas of Section 2, or the corollaries of Section 5. The proofs of Theorems 1 and 2 are essentially unchanged.

REFERENCES