Sectionally complemented chopped lattices

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Conference on Lattice Theory, 2006
N. Funayama and T. Nakayama,
*On the congruence relations on lattices*,
N. Funayama and T. Nakayama,
*On the congruence relations on lattices*,

**Theorem**

The congruence lattice of a lattice is distributive.
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R. P. Dilworth,

**Theorem**

Every finite distributive lattice can be represented as the congruence lattice of a finite lattice.
G. Grätzer and E. T. Schmidt,
*On congruence lattices of lattices*,
(First published proof of the Dilworth Theorem.
Representing with a sectionally complemented lattice.)
I. G. Grätzer and H. Lakser,  
*Characterizing the 1960 sectional complement.*  

II. G. Grätzer and H. Lakser,  
*Generalizing the 1960 sectional complement with an application to congruence restrictions.*  

III. G. Grätzer, H. Lakser, and M. Roddy,  
*The general problem.*  

IV. G. Grätzer and M. Roddy,  
*How far does the Atom Lemma go?*  

http://www.maths.umanitoba.ca/homepages/gratzer.html/
The congruences of a lattice form the congruence lattice. In the past half-century, the study of congruence lattices has become a large and important field with a great number of interesting and deep results and many open problems. This self-contained exposition by one of the leading experts in lattice theory, George Grätzer, presents the major results on congruence lattices of finite lattices featuring the author's signature Proof-by-Picture method and its conversion to transparencies.

Key features:
• Includes the latest findings from a pioneering researcher in the field
• Insightful discussion of techniques to construct "nice" finite lattices with given congruence lattices and "nice" congruence-preserving extensions
• Contains complete proofs, an extensive bibliography and index, and nearly 80 open problems
• Additional information provided by the author online at: http://www.maths.umanitoba.ca/homepages/gratzer.html/

The book is appropriate for a one-semester graduate course in lattice theory, yet is also designed as a practical reference for researchers studying lattices.
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Outline

1 Background

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4 How far does the Atom Lemma go

5 Congruence lattices of ideals
1. **Background**
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   - Ideals
   - Congruences
   - The 1960 construction
   - The 1960 sectional complement

2. **Characterizing the 1960 sectional complement**

3. **The general problem**

4. **How far does the Atom Lemma go**

5. **Congruence lattices of ideals**
A \textit{finite} meet-semilattice $\langle M, \land \rangle$ may be regarded as a partial algebra, $\langle M, \land, \lor \rangle$, called a \textit{chopped lattice}, where $\land$ is an \textbf{operation}, $\lor$ is a \textbf{partial operation} (the least upper bound of $a$ and $b$, provided that it exists).
A finite meet-semilattice \( \langle M, \land \rangle \) may be regarded as a partial algebra, \( \langle M, \land, \lor \rangle \), called a chopped lattice, where \( \land \) is an operation, \( \lor \) is a partial operation (the least upper bound of \( a \) and \( b \), provided that it exists).
We can obtain a chopped lattice by taking a finite lattice $L$ with unit, 1,
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We can obtain a chopped lattice by taking a finite lattice $L$ with unit, 1, and defining $M = L - \{1\}$.

The converse also holds: by adding a new unit 1 to a chopped lattice $M$, we obtain a finite lattice $L$. 
A more useful example is obtained with merging.
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$$J = C \cap D$$

is an ideal in both $C$ and $D$. 

![Diagram of merging](image)
A more useful example is obtained with merging. Let $C$ and $D$ be lattices such that

$$J = C \cap D$$

is an ideal in both $C$ and $D$. Then, with the natural ordering,

$$C \cup D,$$

called the merging of $C$ and $D$, is a chopped lattice.
A (chopped) lattice $M$ is **sectionally complemented** if for all $a \leq b$ in $M$, there is a $c \in M$ satisfying $a \land c = 0$ and $a \lor c = b$. 

A (chopped) lattice $M$ is **sectionally complemented** if for all $a \leq b$ in $M$, there is a $c \in M$ satisfying $a \land c = 0$ and $a \lor c = b$. 

![Diagram](image-url)
A nonempty subset $I$ of a chopped lattice $M$ is an ideal iff
it is a down-set with the property:

(Id) $a, b \in I$ implies that $a \vee b \in I$, provided that $a \vee b$ exists.
A nonempty subset $I$ of a chopped lattice $M$ is an ideal iff it is a down-set with the property:

(Id) $a, b \in I$ implies that $a \lor b \in I$, provided that $a \lor b$ exists.

The set $\text{Id} M$ of all ideals of $M$ is an order under set inclusion; as an order, it is a lattice.
Ideal examples

Ideals: blue; blue and red.
Ideal examples

Ideals: blue; blue and red.

Not an ideal: reds.
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imply that

(SP\$\land\$) \quad a \land c \equiv b \land d \ (\Theta),

(SP\$\lor\$) \quad a \lor c \equiv b \lor d \ (\Theta).
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a \equiv b \ (\Theta) \ \text{and} \ c \equiv d \ (\Theta)\]
imply that 
\[
\begin{align*}
(SP_\wedge) & \quad a \wedge c \equiv b \wedge d \ (\Theta), \\
(SP_\vee) & \quad a \vee c \equiv b \vee d \ (\Theta).
\end{align*}
\]

Define an equivalence relation $\Theta$ to be a congruence of a chopped lattice $M$ as we defined it for lattices, except that we require $SP_\vee$ only when both joins exist.
An equivalence relation $\Theta$ on a lattice $L$ is called a **congruence relation** (congruence) of $L$

iff

\[ a \equiv b \ (\Theta) \text{ and } c \equiv d \ (\Theta) \]

imply that

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\text{(SP}_\wedge\text{)} & \quad a \wedge c \equiv b \wedge d \ (\Theta), \\
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Define an equivalence relation $\Theta$ to be a **congruence** of a chopped lattice $M$

as we defined it for lattices,

except that we require $\text{SP}_\lor$ only when both joins exist.

The set $\text{Con}_M$ of all congruence relations of $M$,

ordered by set inclusion, is a (distributive) lattice.
Congruence (two classes)
The map $m \mapsto \{ x \mid x \leq m \}$ embeds the chopped lattice $M$ into the lattice $\text{Id} M$, so we can regard $\text{Id} M$ as an extension.

**Lemma (Grätzer-Lakser 1968)**

Let $M$ be a chopped lattice. Then every congruence of $M$ has exactly one extension to a congruence of $\text{Id} M$. So every chopped lattice $M$ has a lattice extension $L$ with the same congruence lattice. $L$ is a congruence preserving extension of $M$. 

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So every chopped lattice $M$ has a lattice extension $L$ with the same congruence lattice. $L$ is a congruence preserving extension of $M$. 
The extension destroys all nontrivial congruences.

The bad
The bad

The extension destroys all nontrivial congruences.
Illustration

And the good
And the good

The ideal lattice;
all congruences extend uniquely.
The gadget

For a finite order $P$, by merging together copies of $N_6$, we get a sectionally complemented chopped lattice $M$. 

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p1 \quad p(q) \quad q2\quad q1
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p1 \quad q \quad q1
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For a finite order $P$, by merging together copies of $N_6$, we get a sectionally complemented chopped lattice $M$. 
Example 1: The three-element chain.
Example 2: The three-element order $P_V$. 
Example 3: The three-element order $P_H$. 

![Diagram of the three-element order $P_H$.]
For a finite order $P$, merge copies of $N_6$ together as in the examples, to obtain a sectionally complemented chopped lattice $M$ such that the ordered set of join-irreducibles of $M$ is isomorphic to $P$. 

The 1960 construction
For a finite order $P$, merge copies of $N_6$ together as in the examples, to obtain a sectionally complemented chopped lattice $M$ such that the ordered set of join-irreducibles of $M$ is isomorphic to $P$.

This is the 1960 construction.
Theorem (Grätzer-Schmidt 1960)

Let $M$ be the chopped lattice constructed from a finite order $P$ by merging. Then the ideal lattice of $M$ is a sectionally complemented lattice.
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Let $M$ be the chopped lattice constructed from a finite order $P$ by merging. Then the ideal lattice of $M$ is a sectionally complemented lattice.

We will call this result The 1960 Theorem.
Let $I \subseteq J$ be ideals of the chopped lattice $M$. 

Defining a sectional complement
Let $I \subseteq J$ be ideals of the chopped lattice $M$. Let $S = S(I, J)$ be the ideal generated by all the atoms of $J - I$.Obviously, $I \lor S = J$.
Let \( I \subseteq J \) be ideals of the chopped lattice \( M \).
Let \( S = S(I, J) \) be the ideal
\[
generated by all the atoms of \( J - I \).
\]
Obviously, \( I \vee S = J \).
Let $I \subseteq J$ be ideals of the chopped lattice $M$.
Let $S = S(I, J)$ be the ideal generated by all the atoms of $J - I$.
Obviously, $I \vee S = J$.

$J = M$ and $I$ reds, atoms of $J - I$: yellows.
$I \wedge S = \{0\}$ fails: $q_1 \in I \wedge S$. 
Let $I \subseteq J$ be ideals of the chopped lattice $M$. 
Let $I \subseteq J$ be ideals of the chopped lattice $M$. Then $q \in P$ splits over $\langle I, J \rangle$, if there are $p \succ q$ in $P$, with $p_1, q_i \in J - I$ (yellow) and $q_{i+1} \in I$ (red).
Let \( I \subseteq J \) be ideals of the chopped lattice \( M \).
Then \( q \in P \) splits over \( \langle I, J \rangle \),
if there are \( p \succ q \) in \( P \), with \( p_1, q_i \in J - I \) (yellow) and \( q_{i+1} \in I \) (red).

Let \( S = S(I, J) \) be the ideal
generated by all the atoms of \( J - I \) that do not split.
Theorem (Grätzer-Schmidt 1960; stronger form)

Let $M$ be the chopped lattice constructed from a finite order $P$ by merging. Let $I \subseteq J$ be ideals of $M$. Then $S(I, J)$ is a sectional complement of $I$ in $J$. 

The 1960 sectional complement
Theorem (Grätzer-Schmidt 1960; stronger form)

Let $M$ be the chopped lattice constructed from a finite order $P$ by merging.
Let $I \subseteq J$ be ideals of $M$.
Then $S(I, J)$ is a sectional complement of $I$ in $J$.

We call $S(I, J)$ the 1960 sectional complement.
1 Background

2 Characterizing the 1960 sectional complement
   • What it is not
   • The characterization theorem

3 The general problem

4 How far does the Atom Lemma go

5 Congruence lattices of ideals
If $P$ is the two-element chain, then $M$ is

![Diagram of a lattice with elements $p_1$, $q_2$, $q$, $p(q)$, $q_1$, and $0$.]
If $P$ is the two-element chain, then $M$ is

If $I = \text{id}(p_1)$ and $J = M$, then the 1960 sectional complement is $\text{id}(q)$;
If $P$ is the two-element chain, then $M$ is

If $I = \text{id}(p_1)$ and $J = M$, then the 1960 sectional complement is $\text{id}(q)$; $\text{id}(q_1)$ is a smaller sectional complement,
If $P$ is the two-element chain, then $M$ is

If $I = \text{id}(p_1)$ and $J = M$, then the 1960 sectional complement is $\text{id}(q)$; $\text{id}(q_1)$ is a smaller sectional complement, so the 1960 sectional complement is not minimal.
If $P$ is the three-element chain, then $M$ is

\[
\begin{array}{c}
P \\
\text{a} \\
\text{b} \\
\text{c}
\end{array}
\quad
\begin{array}{c}
M \\
a(b) \\
b \\
b(c) \\
0 \\
\text{a}_1 \\
\text{b}_2 \\
\text{b}_1 \\
\text{c}_2 \\
\text{c}_1
\end{array}
\]
If $P$ is the three-element chain, then $M$ is

If $I = \{b_2, c_2, 0\}$ (the black-filled elements) and $J = M$, then the 1960 sectional complement is $id(M)$. 

$\{a_1, c_1, 0\}$ is a bigger sectional complement.
If $P$ is the three-element chain, then $M$ is

$$M = b(c)$$

$$P = a \quad b \quad c$$

If $I = \{b_2, c_2, 0\}$ (the black-filled elements) and $J = M$, then the 1960 sectional complement is $\text{id}(a_1)$;
If \( P \) is the three-element chain, then \( M \) is

\[
\begin{align*}
&\begin{array}{c}
& a \\
& b \\
& c \\
\end{array}
\quad \begin{array}{c}
& a(b) \\
& N(a, b) \\
& b \\
& b(c) \\
& N(b, c) \\
& c \\
\end{array}
\quad \begin{array}{c}
& a_1 \\
& b_2 \\
& b_1 \\
& c_2 \\
& c_1 \\
\end{array}
\quad 0
\end{align*}
\]

If \( I = \{b_2, c_2, 0\} \) (the black-filled elements) and \( J = M \), then the 1960 sectional complement is \( \text{id}(a_1); \) \( \{a_1, c_1, 0\} \) is a bigger sectional complement.
Let $K$ be a finite lattice and let $a \in K$.
A complement $s$ of $a$ is a strong complement of $a$, if $s \land x$ is a complement of $a$ whenever $x$ is.
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A complement $s$ of $a$ is a strong complement of $a$,
if $s \land x$ is a complement of $a$ whenever $x$ is.

**Theorem (Grätzer-Lakser 2005)**

The 1960 sectional complement is
the smallest strong sectional complement.
1 Background

2 Characterizing the 1960 sectional complement

3 The general problem
   - The general problem
   - The Atom Lemma
   - A small example
   - A cyclic example

4 How far does the Atom Lemma go

5 Congruence lattices of ideals
In the 1960 construction,
we get sectionally complemented chopped lattices such as
The 1960 Theorem

In the 1960 construction, we get sectionally complemented chopped lattices such as

![Diagram of a sectionally complemented chopped lattice]

The 1960 Theorem states that the ideal lattices of these sectionally complemented chopped lattices are again sectionally complemented.
**Problem**

Let $M$ be a sectionally complemented chopped lattice. Under what conditions is $\text{Id} M$ sectionally complemented?
Problem

Let $M$ be a sectionally complemented chopped lattice. Under what conditions is $\text{Id} M$ sectionally complemented?

Theorem (Harry Lakser)

Let $M$ be a sectionally complemented chopped lattice. Then $\text{Id} M$ is sectionally complemented iff it is sectionally complemented.
**Problem**

Let $M$ be a sectionally complemented chopped lattice. Under what conditions is $\mathbb{Id} M$ sectionally complemented?

**Theorem (Harry Lakser)**

Let $M$ be a sectionally complemented chopped lattice. Then $\mathbb{Id} M$ is sectionally complemented iff it is sectionally complemented.

:-)
Lemma (The Atom Lemma; Grätzer-Schmidt 1999)

Let $A$ and $B$ be finite sectionally complemented lattices such that $J = A \cap B$ is an ideal in both $A$ and $B$. Let $M$ be the chopped lattice we obtain by merging $A$ and $B$. If $|J| \leq 2$, then $\text{Id } M$ is a sectionally complemented lattice.
**The Atom Lemma**

**Lemma (The Atom Lemma; Grätzer-Schmidt 1999)**

Let $A$ and $B$ be finite sectionally complemented lattices such that $J = A \cap B$ is an ideal in both $A$ and $B$. Let $M$ be the chopped lattice we obtain by merging $A$ and $B$. If $|J| \leq 2$, then $\text{Id} M$ is a sectionally complemented lattice.

If $|J| = 1$, that is, $J = \{0\}$, then the Atom Lemma is trivial.
Lemma (The Atom Lemma; Grätzer-Schmidt 1999)

Let $A$ and $B$ be finite sectionally complemented lattices such that $J = A \cap B$ is an ideal in both $A$ and $B$. Let $M$ be the chopped lattice we obtain by merging $A$ and $B$. If $|J| \leq 2$, then $\text{Id } M$ is a sectionally complemented lattice.

If $|J| = 1$, that is, $J = \{0\}$, then the Atom Lemma is trivial. If $|J| = 2$, then $J = \{0, p\}$, where $p$ is an atom; this gives the name of the lemma.
The Atom Lemma illustrated
Grätzer-Lakser-Roddy 2005:
Grätzer-Lakser-Roddy 2005:

The three-element ideal of black-filled elements has no sectional complement in $M$. 
Identify the zeros, the two red-, the two blue-, and the two yellow-filled atoms, to obtain the chopped lattice $M$.

In $M$, $m_1 \land m_2$ (blue), $m_2 \land m_3$ (red), $m_3 \land m_1$ (yellow) are atoms.

The four-element ideal of black-filled elements has no complement in $M$; so the Atom Lemma fails for a three-cycle.
Identify the zeros, the two red-, the two blue-, and the two yellow-filled atoms, to obtain the chopped lattice $M$. 
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In $M$, $m_1 \wedge m_2$ (blue), $m_2 \wedge m_3$ (red), $m_3 \wedge m_1$ (yellow) are atoms.
Identify the zeros, the two red-, the two blue-, and the two yellow-filled atoms, to obtain the chopped lattice $M$.

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In $M$, $m_1 \wedge m_2$ (blue), $m_2 \wedge m_3$ (red), $m_3 \wedge m_1$ (yellow) are atoms. The four-element ideal of black-filled elements has no complement in $M$; so the Atom Lemma fails for a three-cycle.
1. Background

2. Characterizing the 1960 sectional complement

3. The general problem

4. How far does the Atom Lemma go
   - Proving the Atom Lemma
   - How far does the method of the Atom Lemma go

5. Congruence lattices of ideals
So let $A$ and $B$ be sectionally complemented chopped lattices. We identify the zero and the atom $p$ to obtain $M$. We want to show that the ideal $I = \text{id}(i_1) \cup \text{id}(i_2)$ has a complement in $M$. 

![Diagram of a lattice with nodes labeled $m_1$, $m_2$, $i_1$, $i_2$, $p$, and $A$ and $B$.]
So let $A$ and $B$ be sectionally complemented chopped lattices. We identify the zero and the atom $p$ to obtain $M$. 
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$$I = \text{id}(i_1) \cup \text{id}(i_2)$$

has a complement in $M$. 

[Diagram of a lattice with vertices labeled $m_1$, $m_2$, $i_1$, $i_2$, $0$, $p$, $A$, and $B$.]
Let $s_1$ be the sectional complement of $i_1$ in $m_1$. 
Let $s_1$ be the sectional complement of $i_1$ in $m_1$. Let $s_2$ be the sectional complement of $i_2$ in $m_2$.
Let \( s_1 \) be the sectional complement of \( i_1 \) in \( m_1 \).
Let \( s_2 \) be the sectional complement of \( i_2 \) in \( m_2 \).
If \( S = \text{id}(s_1) \cup \text{id}(s_2) \) is an ideal,
then it is a sectional complement of \( I \) in \( M \).
If $S = \text{id}(s_1) \cup \text{id}(s_2)$ is not an ideal, say, $p \land s_1 = 0$ and $p \leq s_2$, then let $s'_2$ be a sectional complement of $p$ in $s_2$. Then $S' = \text{id}(s_1) \cup \text{id}(s'_2)$ is an ideal, a sectional complement of $I$ in $M$. 
The method of the proof

To show that \( I = \text{id}(i_1) \cup \text{id}(i_2) \) has a complement in \( M \), form sectional complements “componentwise” to obtain \( S \). If \( S \) is an ideal, we are done.
If not, we adjust—lower—one component to obtain \( S' \), an ideal.
We have to make sure that \( S' \) is big enough.
We have two results of the same type:
the 1960 Theorem and the Atom Lemma,
both proving that
the ideal lattice of a sectionally complemented chopped lattice is
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We have two results of the same type: the 1960 Theorem and the Atom Lemma, both proving that*

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**Question:** Can we use the method of the proof of the Atom Lemma (form sectional complements “componentwise” and downward adjust) to prove the 1960 Theorem?
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Let $I \subseteq J$ be ideals of $M$;
let $s_{p,q}$ be the largest sectional complement of $i_{p,q}$ below $j_{p,q}$ (in $N(p,q)$);
let $S$ be the down-set generated by the $s_{p,q}$.
We obtain $S^*$ by applying an algorithm to $S$. 
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Theorem

$S^*$ is an ideal, a sectional complement of $I$ in $J$ in $Id M$.
Moreover, for every $p \succ q$ in $P$, we have $s_{p,q}^* = s_{p,q}$ or $s_{p,q}^* \prec s_{p,q}$.
1 Background

2 Characterizing the 1960 sectional complement

3 The general problem

4 How far does the Atom Lemma go

5 Congruence lattices of ideals
   - Background
   - The result
The Dilworth Theorem

**Theorem**

Let $D$ be a finite distributive lattice. Then there exists a finite lattice $L$ such that $\text{Con} L$ is isomorphic to $D$. 
If $I$ is an ideal of the lattice $L$, then the restriction map

$$\text{Con } L \rightarrow \text{Con } I$$

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**Theorem**

Let $D$ and $E$ be finite distributive lattices; let $D$ be nontrivial. Let $\varphi$ be a $\{0, 1\}$-preserving homomorphism of $D$ into $E$. Then there exists a finite lattice $L$ and an ideal $I$ of $L$ such that

$$D \cong \text{Con } L, \quad E \cong \text{Con } I,$$

and $\varphi$ is represented by the restriction map.
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If $I$ is an ideal of the lattice $L$, then the restriction map

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Here is the corresponding representation theorem for sectionally complemented finite lattices (Grätzer-Lakser 2005):
If $I$ is an ideal of the lattice $L$, then the restriction map

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is a \( \{0, 1\} \)-preserving homomorphism.

Here is the corresponding representation theorem for sectionally complemented finite lattices (Grätzer-Lakser 2005):

**Theorem**

Let $D$ and $E$ be finite distributive lattices; let $D$ be nontrivial. Let $\varphi$ be a \( \{0, 1\} \)-preserving homomorphism of $D$ into $E$. Then there exists a **sectionally complemented** finite lattice $L$ and an ideal $I$ of $L$ such that

$$D \cong \text{Con } L, \quad E \cong \text{Con } I,$$

and $\varphi$ is represented by the restriction map.